Lecture Notes on Algebraic K-Theory

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Contents

1 Introduction

These are the notes for the lecture course on algebraic K–theory held at the Max Planck Institute for Mathematics in Bonn in the summer term (April–July) of 2024. The aim of the course is to define and present the fundamental theorems of algebraic K–theory from the ∞–categorical viewpoint, as well as to take stock of recent landmark advances in the subject. The course begins from the premise that algebraic K–theory is an interesting and deep subject in and of itself, and thus will not cover its myriad relations to other parts of pure mathematics.

The prerequisites for this course are a good understanding of basic algebraic topology, ring theory, and (ordinary) category theory. Familiarity with ∞–categories is not assumed, although it is certainly very helpful. Most of the foundational materials on this will be recounted in Chapter 2 albeit at a brisk pace.

1.1 K–theory as a universal homology theory

There is a procedure $|-|$ which takes a finite set and outputs its cardinality, and it is a well–known mathematical fact that, for finite subsets $S, T \subseteq U$, we have the inclusion–exclusion (or excision) principle for cardinalities

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Note that in light of this excision principle, the procedure $|-|$ is totally determined by the requirement that $|\{*\}| = 1$. A large swath of the field of higher algebra may arguably be said to study generalisations of such excisions to the homotopical context, and the appropriate notion is that of a *homology theory*, i.e. a functor

$$H \colon \{ \text{ finite spaces } \} \to \{ \text{ some “algebraic” category } \}$$

such that for all (homotopy) pushouts $A \cup_C B$ in finite spaces, $H(A \cup_C B) \simeq H(A) \cup_{H(C)} H(B)$. As in the case of the procedure $|-|$, since all finite spaces can be built as a (homotopy) pushout from the point $*$, the homology theory H is totally determined by the object $H(*)$. From this philosophical excursion, we learn that, (very!) loosely speaking, a homology theory is a counting or quantifying procedure satisfying some form of excision, the specific form of which may be pinned down by some specified requirements on what or how we are counting.

One way to summarise what K–theory is about is the following: it is a method of quantifying interesting algebraic and geometric mathematical objects by going beyond plain numbers and counting with higher algebra. The payoff of using counting systems more sophisticated than numbers is that they have better formal properties that can be employed to deduce new calculations from old ones.

As we shall see in this course, a theorem of Barwick and Blumberg–Gepner–Tabuada tells us that the algebraic K–theory functor $\mathrm{K}(-)$ is the universal homology theory on stable categories associated to the moduli space functor

$$(-)^{\simeq}\colon \mathrm{Cat}^{\mathrm{ex}} \longrightarrow \mathrm{An}$$

where $(-)^{\simeq}$ takes a small stable category $\mathcal{C}$ and outputs the space of objects $\mathcal{C}^{\simeq}$ in $\mathcal{C}$. It will be universal in the sense that it is the initial functor receiving a natural transformation $(-)^{\simeq} \Rightarrow \mathrm{K}(-)$ satisfying excision with respect to "exact sequences" of stable categories. This theme, and many more, will be the subject of additivity and localisation which we will cover in Chapter 4. Furthermore, there is also a new K–theoretic excision discovered relatively recently by Land–Tamme which is of fundamental importance and this will be discussed in Chapter 6.

While the above–mentioned excision properties enjoyed by $\mathrm{K}(-)$ are highly desirable in that provides the principle needed to glue known calculations together, it cannot go as far by itself because we would need to have known some basic calculations to begin with. Fortunately, there is a slew of theorems which guarantees that some of the algebraic K–theory spectra are equivalently given by other K–theoretic variants which are more computable from first principles. Stating these results and proving some of them using Grothendieck's important technique of dévissage will be the subject of Chapter 5. One such variant will be the so–called *group–completion K–theory* that will be studied in Chapter 3. The advantage of this variant is that their homology groups are often "computable" in that the group–completion Theorem 3.1.14 allows us to port results from the industry of group homologies and homological stability to help compute their homologies. In fact, the first major calculation in algebraic K–theory which was given by Quillen proceeded precisely in this manner, augmented by some very clever ideas of his.

2 Recollections on ∞-engineering

We begin these notes by gathering some basics from the theory of higher categories. As far as possible, our presentation will be done in a model–independent way. The purpose of this section is merely to provide a convenient reference point for the reader not so familiar with ∞–categories. As such, it will mostly be just a list of standard results and informal definitions that we need in the main body of this document, with precise references where possible, without it being woven into a cogent story.

As a matter of psychological reassurance, we would go so far as to claim that the proper rules of engagement with ∞–categories is more or less the same as those with classical 1–categories, and so it might be helpful to read most of these recollections in the psychologically more comforting setting of 1–categories. In some sense, the basic philosophy of this course is that a purely axiomatic understanding of higher categories - to the extent of having good symbolic intuition and being able to manipulate the symbols correctly without having to resort to specific models of ∞–categories - is sufficient (and often even desirable!) for the purposes of working in stable homotopy theory. In short, these notes are meant to pave the way for the reader to become an ∞–engineer[1].

Finally, a word of warning: since these results are so embedded in the canon, it is a bit difficult to make the correct attribution as to where the result first appeared. As such, many of these references will be pointing to *a* place where it appeared without the implicit claim about origins. Of course, the most biblically comprehensive sources for all things ∞–categorical are Lurie's pair of tomes [Lur12; Lur17], but some good (and shorter) one–stop locations for many of the basic results are the excellent set of lecture notes by Fabian Hebestreit, as expanded by Ferdinand Wagner [HW21], as well as the survey by David Gepner [Gep19].

2.1 Basic setup

The informal stage

Roughly speaking, ∞–categories may be thought of as categories enriched in spaces (whatever "spaces" mean), that is, they are certain category–like mathematical objects for which there is a whole space's worth of morphisms between two objects rather than just a set's worth of them. Unfortunately, there is always a chicken–and–egg

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[1] We thank Vignesh Subramanian for coining the phrase ∞–engineer.

problem in defining ∞–categories and spaces precisely if one wants to exposit model–independently. Therefore, we will embrace the sins of ouroboros and just proceed to describe these things in more detail very informally and circularly. This is neither a self–contained, fully precise account of higher category theory, nor was it meant to be so.

Notation 2.1.1 (Anima)**.** We write An for the ∞–category of spaces/∞–groupoids/anima/homotopy types. It is the ∞–category freely generated by the point under small (∞–)colimits (the precise meaning of which we shall explain explain below). For the reader acquainted with the classical literature, this is the ∞–category associated to Kan complexes/CW complexes/topological spaces under the appropriate model structures. Informally, An should be thought of as the ∞–category generated by the point under coproducts and (homotopy) pushouts. Indeed, it is the extra adjective "homotopy" in homotopy pushouts that sets it apart from the category of sets. In this sense, anima should be viewed as the appropriate homotopical analogue of sets which allows one to move from the classical categorical setting to the higher categorical one.

Notation 2.1.2. We write Cat and Cat_∞ interchangeably for the ∞–category of small ∞–categories. We also write Cat_1 for the ∞–category of 1–categories (i.e. ordinary, classical, pre–grad–school categories). As we will summarise in Fact 2.1.4, Cat_1 may naturally be viewed as a full subcategory of Cat consisting of those ∞–categories $\mathcal{C}$ with the property that for any two objects $x, y \in \mathcal{C}$, the anima $\mathrm{Map}_{\mathcal{C}}(x,y)$ is 0–truncated, i.e. for any $f \in \pi_0 \mathrm{Map}_{\mathcal{C}}(x,y)$, we have $\pi_i\big(\mathrm{Map}_{\mathcal{C}}(x,y), f\big) = 0$ for $i \geq 1$.

Warning 2.1.3. There is a big difference between the ∞–category of 1–categories Cat_1 and the 1–category of 1–categories $\mathrm{Cat}_1^{(1)}$. The morphism *set* in $\mathrm{Cat}_1^{(1)}$ consists of the set of functors between 1–categories, whereas the morphism *anima* in Cat_1 consists of the groupoid of functors between 1–categories and natural isomorphisms between functors. In higher category theory, we will only consider Cat_1 since $\mathrm{Cat}_1^{(1)}$ belongs to a collection of "evil" concepts.

Taking for granted for the moment the notion of adjunctions and (co)limits which we shall explain soon below, all of these may be organised succinctly as follows:

Fact 2.1.4 (Cosmic adjunctions). By virtue of the so–called *Grothendieck homotopy hypothesis*, any valid theory of higher categories should in particular admit the result that spaces are equivalent to ∞–groupoids (ie. ∞–categories where all morphisms are equivalences); moreover, 1–categories should be an instance of an ∞–category. From this, we have the adjunctions (where a left adjoint is written on top of its right adjoint)

$$\begin{tikzcd}
\mathrm{Cat}_1 \arrow[r, hook, "N"'] & \mathrm{Cat}_\infty \arrow[l, bend right, "\mathrm{Ho}"'] \arrow[r, bend left, "|-|"] \arrow[r, bend right, "(-)^{\simeq}"'] & \mathrm{An} \arrow[l, hook']
\end{tikzcd}$$

where Ho is taking the homotopy 1–category of an ∞–category (ie. taking the connected components of mapping spaces in an ∞–category); N is viewing a 1–category as an ∞–category, where the notation N is supposed to remind the quasi–categorically–minded reader of taking the nerve simplicial set of a 1–category; $|-|$ is the functor which inverts all morphisms in an ∞–category; the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ is viewing spaces as ∞–groupoids; and finally $(-)^{\simeq}$ is taking the so–called *core groupoid* of an ∞–category, i.e. remembering only the morphisms which are equivalences.

Importantly from these adjunctions, we see that the inclusion $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ preserves limits, and the inclusion $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ preserves both limits and colimits. In other words, taking limits of 1–categories in Cat_∞ still yields a 1–category; and taking (co)limits of spaces in Cat_∞ still yields a space.

Fact 2.1.5. In An, filtered colimits commute with finite limits. Moreover, geometric realisations (ie. colimits indexed by the 1–category $\mathbf{\Delta}^{\mathrm{op}}$), commute with finite products.

As alluded to above, one of the many structures that an ∞–category encodes is that of a mapping space/anima between two objects, which may be precisely extracted as follows:

Construction 2.1.6 (Mapping anima)**.** For objects x, y in an ∞–category $\mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x,y) \in \mathrm{An}$ is defined as the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d, "{(s,t)}"] \\
* \arrow[r, "{(x,y)}"'] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

Fact 2.1.7. For $X \in \mathrm{An}$, there is a formula for the mapping space of $\mathrm{Fun}(X, \mathcal{C})$ given by the following: for $\varphi, \psi \in \mathrm{Fun}(X, \mathcal{C})$, we have

$$\mathrm{Map}_{\mathrm{Fun}(X,\mathcal{C})}(\varphi, \psi) \simeq \lim_{x \in X} \mathrm{Map}_{\mathcal{C}}(\varphi(x), \psi(x))$$

Terminology 2.1.8. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is said to be *fully faithful (resp. faithful)* if for all $x, y \in \mathcal{C}$, the induced map

$$\mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$$

is an equivalence (resp. inclusion of path components) of anima. Be warned that unlike in the land of classical categories, in general no notion of a "full" functor is well–behaved and so they will never show up. As another warning, note that an anima is strictly less structure than a topological space, and so it will also not make sense to speak of fully faithfulness as the inclusion of a subspace (e.g. a positive codimension submanifold) in the sense of point–set topology.

Fact 2.1.9. For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_\infty$, we may construct the ∞–category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ from $\mathcal{C}$ to $\mathcal{D}$ whose core groupoid is the mapping anima between $\mathcal{C}$ and $\mathcal{D}$, i.e. we have a canonical equivalence in An

$$\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$$

Adjunctions and (co)limits

Definition 2.1.10. Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors. An *adjunction datum* $L \dashv R$ *between* L *and* R consists of a natural equivalence of functors

$$\mathrm{Map}_{\mathcal{C}}(-, R-) \simeq \mathrm{Map}_{\mathcal{D}}(L-, -)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathrm{An}$$

Given such a datum, we will say that L is left adjoint to R (or conversely, R is right adjoint to L).

Warning 2.1.11. It is just a *property* for a functor to admit a left or a right adjoint. But an adjunction datum is a *structure*, i.e. for two fixed functors L and R, there can be many inequivalent adjunction data witnessing that L is left adjoint to R!

Notation 2.1.12. In this document, we always write left adjoints on top of its right adjoint. In other words, when we write $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : R$ or

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

we mean that L is the left adjoint to R.

Definition 2.1.13. An adjunction $L \dashv R$ is a *Bousfield localisation (resp. Bousfield colocalisation)* if the right adjoint R (resp. the left adjoint L) is fully faithful.

Observation 2.1.14. It is a straightforward exercise to see that the condition of being a Bousfield localisation (resp. colocalisation) is equivalent to the condition that the adjunction counit (resp. adjunction unit) is an equivalence.

Observation 2.1.15. It is sometimes useful to observe that a Bousfield localisation is an equivalence if and only if the left adjoint L is conservative. Proving this is a simple and instructive exercise.

Definition 2.1.16. Let $p\colon I \to *$ be the unique functor and $\mathcal{C}$ a category. If it exists, the left adjoint $p_!$ (resp. right adjoint p_*) to the functor $p^*\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is called the colimit colim_I of I (resp. limit $\lim_I$ of I) functor.

Fact 2.1.17. Let $f\colon I \to J$ be a functor and $\mathcal{C}$ an ∞–category with all small (co)limits. The left adjoint $f_!$ (resp. right adjoint f_*) of the restriction functor $f^*\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ are called the *left Kan extension (resp. right Kan extension) along* f. If f is fully faithful, then $f_!$ and f_* are fully faithful too.

Definition 2.1.18 (Adjoint objects)**.** Let $G\colon \mathcal{D} \to \mathcal{C}$ be a functor, $x \in \mathcal{C}$, $y \in \mathcal{D}$, and $\eta : x \to G(y)$. We say that η *witnesses y as a left adjoint object to x under G* if

$$\mathrm{Map}_{\mathcal{D}}(y, -) \xrightarrow{G} \mathrm{Map}_{\mathcal{C}}(Gy, G-) \xrightarrow{\eta^*} \mathrm{Map}_{\mathcal{C}}(x, G-)$$

is an equivalence of functors $\mathcal{D} \to \mathrm{An}$.

Proposition 2.1.19 (Omnibus adjunctions)**.** *Let $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : R$ be an adjunction and $\mathcal{E}, J$ be some other categories. Write $p\colon \mathcal{C} \to *$ and $q\colon \mathcal{D} \to *$ for the unique maps.*

1. *The adjunction $L \dashv R$ induces adjunctions*

$$R^*\colon\ \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \rightleftharpoons \mathrm{Fun}(\mathcal{C}, \mathcal{E}) : L^* \qquad\qquad L_*\colon\ \mathrm{Fun}(J, \mathcal{C}) \rightleftharpoons \mathrm{Fun}(J, \mathcal{D}) : R_*.$$

   If $L \dashv R$ was a Bousfield localisation, then so are the adjunctions $R^* \dashv L^*$ and $L_* \dashv R_*$,

2. *(Special case of Quillen's Theorem A): The functor L is coinitial and R is cofinal, i.e. for $\partial \in \mathrm{Fun}(J, \mathcal{C})$ and $\delta \in \mathrm{Fun}(J, \mathcal{D})$, the canonical maps*

$$q_* \longrightarrow p_* L^* \qquad\qquad q_! R^* \longrightarrow p_!$$

   are equivalences. That is, restriction along left adjoints (resp. right adjoints) preserve limits (resp. colimits),

3. *(Objectwise construction of adjoints): $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$ if and only if all objects in $\mathcal{C}$ admits a left adjoint object.*

Proof. Point (1) is a straightforward exercise using that adjunctions may equivalently be characterised in terms of the usual triangle identities. Point (2) is a bit messy to prove in full detail since various canonical transformations need to be identified, but we can at least (instructively) show that there are equivalences $q_* \simeq p_* L^*$ and $q_! R^* \simeq p_!$. For example, for the first case, note that $pR \simeq q$ and that $L^* \simeq R_*$ by point (1). Hence, we obtain

$$p_* L^* \simeq p_* R_* \simeq (pR)_* \simeq q_*$$

as required. Finally, for part (3), the trick is to use the Yoneda lemma to help us assemble the various left adjoint objects into a coherent functor. We consider $\mathrm{Map}_{\mathcal{C}}(-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$ as a functor $H : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{D}, \mathrm{An})$. Hence, by hypothesis, the bottom left composition lands in the essential image of the Yoneda embedding (c.f. Theorem 2.1.25) and so we obtain a lift L^{op} in the commuting square

$$\begin{tikzcd}
& \mathcal{D}^{\mathrm{op}} \arrow[d, hook, "y"] \\
\mathcal{C}^{\mathrm{op}} \arrow[ur, dashed, "F^{\mathrm{op}}"] \arrow[r, "H"] & \mathrm{Fun}(\mathcal{D}, \mathrm{An})
\end{tikzcd}$$

This gives a left adjoint since, by construction, $y \circ L^{\mathrm{op}} \simeq H$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{D}, \mathrm{An}))$, and hence $\mathrm{Map}_{\mathcal{D}}(L-, -) \simeq \mathrm{Map}_{\mathcal{C}}(-, G-)$ in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{An})$. □

Terminology 2.1.20. Let $\alpha\colon I \to J$ be a functor. We say that it is *colimit cofinal* if for all functors $F\colon J \to \mathcal{C}$, the canonical map

$$\operatorname*{colim}_I F \circ \alpha \longrightarrow \operatorname*{colim}_J F$$

is an equivalence.

Theorem 2.1.21 (Quillen's Theorem A, [HW21, Thm. I.43])**.** *Let $\alpha\colon I \to J$ be a functor. Then α is colimit cofinal if and only if for all $j \in J$, we have $|j \downarrow \alpha| \simeq *$.*

Exercise 2.1.22. Let $\mathcal{C}$ be a cocomplete category and $X \in \mathcal{C}$. Show that:

(1) there is an adjunction $X \sqcup -\colon \mathcal{C} \rightleftharpoons \mathcal{C}_{X/} : \mathrm{fgt}$. In particular, limits in $\mathcal{C}_{X/}$ are computed in $\mathcal{C}$,

(2) for a diagram $F\colon J \to \mathcal{C}_{/X}$, the colimit of J in $\mathcal{C}_{X/}$ is computed as the pushout

$$\begin{tikzcd}
\mathrm{colim}^{\mathcal{C}}_J \mathrm{const}_J X \arrow[r] \arrow[d, "\mathrm{can}"'] & \mathrm{colim}^{\mathcal{C}}_J \mathrm{fgt} \circ F \arrow[d] \arrow[d, phantom, "\ulcorner", shift right=8] \\
X \arrow[r] & \mathrm{colim}^{\mathcal{C}_{X/}}_J F,
\end{tikzcd}$$

(3) the functor $\mathrm{fgt}\colon \mathcal{C}_{X/} \to \mathcal{C}$ preserves contractible colimits, i.e. the colimits indexed on those diagrams J such that $|J| \simeq *$. **Hint:** *use Theorem 2.1.21 and consider the functor $\alpha\colon J \to *$.*

Yoneda and adjoining colimits

Notation 2.1.23. For $\mathcal{C} \in \mathrm{Cat}$, we will write $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ for the *presheaf category*.

Notation 2.1.24. For cocomplete categories $\mathcal{D}, \mathcal{E}$, we write $\mathrm{Fun}^L(\mathcal{D}, \mathcal{E})$ for the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ consisting of functors that preserve small colimits.

Theorem 2.1.25 (Presheaf cocompletion)**.** *The functor $y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) = \mathrm{PSh}(\mathcal{C})$ which takes $x \in \mathcal{C}$ to the functor $\mathrm{Map}_{\mathcal{C}}(-, x)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ is fully faithful. Moreover, for any cocomplete category $\mathcal{E}$, the induced map $y^*\colon \mathrm{Fun}^L(\mathrm{PSh}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is an equivalence.*

Remark 2.1.26. We may read Theorem 2.1.25 as saying that $\mathrm{PSh}(\mathcal{C})$ is the cocomplete category freely generated by $\mathcal{C}$. In particular, setting $\mathcal{C} = *$, we see that $\mathrm{An} = \mathrm{Fun}(*, \mathrm{An})$ is the cocomplete category freely generated by the point $*$, thus providing a precise incarnation of the slogan in Notation 2.1.1.

Definition 2.1.27. An object $X \in \mathcal{C}$ is said to be *compact* if $\mathrm{Map}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathrm{An}$ preserves filtered colimits. More generally, for a regular cardinal κ, X is called κ-compact if $\mathrm{Map}_{\mathcal{C}}(X,-)$ preserves κ-filtered colimits.

Definition 2.1.28. A set of objects $S \subseteq \mathcal{C}$ is said to be *jointly conservative* if any morphism in $\mathcal{C}$ that gets sent to an equivalence under the functor

$$\prod_{X\in S} \mathrm{Map}_{\mathcal{C}}(X,-)\colon \mathcal{C} \longrightarrow \prod_{S} \mathrm{An}$$

is already an equivalence in $\mathcal{C}$.

Example 2.1.29. In An, the singleton set $\{*\}$ is a jointly conservative set of ω–compact object. In Cat, the two element set $\{\Delta^0 = *, \Delta^1\}$ is a jointly conservative set of ω–compact objects.

Exercise 2.1.30. Suppose we have an adjunction $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : R$.

(1) Show that if R preserves κ–filtered colimits, then L preserves κ–compact objects.

(2) Suppose furthermore that $\mathcal{C}$ has a set of jointly conservative κ–compact objects. Then show that if L preserves κ–compact objects, then R preserves κ–filtered colimits.

The following is an important standard result due originally in 1–categories to [MP87, Lem. 1.7.ii]. The ∞–categorical version is well–known and is recorded for example in [CDH+].

Proposition 2.1.31. *Let $\mathcal{C}$ be cocomplete and let $S \subseteq \mathcal{C}$ be a jointly conservative set of κ–compact objects. Then $\mathcal{C}$ is generated under κ–small colimits by the full subcategory spanned by S.*

We give now another standard construction which allows us to "freely adjoint formal κ–filtered colimits" analogous to the presheaf construction above.

Theorem 2.1.32 (Ind–completion)**.** *Let $\mathcal{C}$ be small. Then for each regular cardinal κ, there exists a category $\mathrm{Ind}_\kappa\mathcal{C}$ equipped with a fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Ind}_\kappa\mathcal{C}$ satisfying the following universal property: for every $\mathcal{D}$ admitting κ–filtered colimits, the functor $\mathcal{C} \hookrightarrow \mathrm{Ind}_\kappa\mathcal{C}$ induces an equivalence*

$$\mathrm{Fun}^{\kappa-\mathrm{filt}}(\mathrm{Ind}_\kappa\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where $\mathrm{Fun}^{\kappa-\mathrm{filt}} \subseteq \mathrm{Fun}$ is the full subcategory of κ–filtered colimit–preserving functors.

Proposition 2.1.33 (∞–Makkai–Pitts)**.** *Let κ be a regular cardinal and $\mathcal{C}$ a cocomplete category. Suppose there is a set S of κ–compact objects in $\mathcal{C}$ which is jointly conservative. Then $\mathcal{C}$ is κ–compactly generated, i.e. the canonical map $\mathrm{Ind}_\kappa(\mathcal{C}^\kappa) \to \mathcal{C}$ is an equivalence.*

Localisations

Terminology 2.1.34. We recall the clarifying distinction between *Dwyer-Kan localisations* and *Bousfield localisations* due to [Hin16]. By Dwyer–Kan localisations, we will mean the following: let $\mathcal{C}$ be a category and S a collection of morphisms in $\mathcal{C}$. Suppose now that a category $S^{-1}\mathcal{C}$ exists and is equipped with a map $\mathrm{DK} : \mathcal{C} \to S^{-1}\mathcal{C}$ inducing the equivalence

$$\mathrm{DK}^* : \mathrm{Fun}(S^{-1}\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{S^{-1}}(\mathcal{C}, \mathcal{D})$$

for all categories $\mathcal{D}$, where $\mathrm{Fun}^{S^{-1}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the -full subcategory of functors sending morphisms in S to equivalences. If such a category exists, then it must necessarily be unique, and this is then defined to be the *–Dwyer–Kan localisation of $\mathcal{C}$ with respect to S*.

Recall from Definition 2.1.13 that by *Bousfield localisations*, we mean a adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : i$ where the right adjoint i is fully faithful. Writing Z for the morphisms in $\mathcal{C}$ that get sent to equivalences under L, we may then view $\mathcal{D}$ as precisely the full subcategory of *Z–local objects*, i.e. those $X \in \mathcal{C}$ such that for any morphism $\varphi\colon A \to B$ in Z, the induced map $\varphi^*\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is an equivalence.

Proposition 2.1.35 (Bousfield implies Dwyer–Kan)**.** *Let $\mathcal{C}, L\mathcal{C}$ be categories and $L : \mathcal{C} \rightleftarrows L\mathcal{C} : i$ be a Bousfield localisation. Let S be the collection of morphisms in $\mathcal{C}$ that are sent to equivalences under L. Then the functor L induces an equivalence $L^* : \mathrm{Fun}(L\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{S^{-1}}(\mathcal{C}, \mathcal{D})$ for any category $\mathcal{D}$ so that $L\mathcal{C}$ is a Dwyer-Kan localisation against S.*

Proof. Since $L \dashv i$ was a Bousfield localisation, we know that $i^* : \mathrm{Fun}(L\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathcal{D}) : L^*$ is also a Bousfield localisation by Proposition 2.1.19, and so in particular L^* is fully faithful. The image of L^* also clearly lands in $\mathrm{Fun}^{S^{-1}}(\mathcal{C}, \mathcal{D})$, and so we are left to show essential surjectivity. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor that inverts morphisms in S. We aim to show that $\varphi \Rightarrow \varphi \circ i \circ L$ is an equivalence. Since $L \dashv i$ was a Bousfield localisation, the unit $\eta : \mathrm{id} \Rightarrow i \circ L$ gets sent to an equivalence under L, and so $\eta \in S$. Since φ inverts S by assumption, in particular it inverts η. □

Proposition 2.1.36 ([Lur12, Prop. 5.2.7.4])**.** *Let $\mathcal{C}$ be a category and $L\colon \mathcal{C} \to \mathcal{C}$ a functor with essential image $L\mathcal{C} \subseteq \mathcal{C}$. Suppose there is a natural transformation*

$$\alpha\colon \mathrm{id} \to L$$

of functors $\mathcal{C} \to \mathcal{C}$ such that for every $X \in \mathcal{C}$, the morphisms

$$\alpha_{LX}, L\alpha_X\colon LX \longrightarrow LLX$$

are equivalences. Then the functor $L\colon \mathcal{C} \to L\mathcal{C}$ participates in a Bousfield localisation

$$\mathcal{C} \underset{\longleftarrow}{\overset{L}{\longrightarrow}} L\mathcal{C}$$

Straightening–unstraightening

As with many things, the following is a key idea due to Grothendieck.

Definition 2.1.37. Let $p\colon \mathcal{E} \to \mathcal{C}$ be a functor and $f\colon x \to y$ a morphism in $\mathcal{E}$. We say that it is a *p–cocartesian morphism* if for any $z \in \mathcal{E}$ the canonical commuting square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(y,z) \arrow[r, "p"] \arrow[d, "f^*"'] & \mathrm{Map}_{\mathcal{C}}(py,pz) \arrow[d, "(pf)^*"] \\
\mathrm{Map}_{\mathcal{E}}(x,z) \arrow[r, "p"'] & \mathrm{Map}_{\mathcal{C}}(px,pz)
\end{tikzcd}$$

is a pullback. Writing $p\text{–Cocart}(\mathcal{E}) \subseteq \mathcal{E}^{\Delta^1}$ for the full subcategory of p–cocartesian morphisms, we say that $p\colon \mathcal{E} \to \mathcal{C}$ is a *cocartesian fibration* if the canonical commuting square

$$\begin{tikzcd}
p\text{–Cocart}(\mathcal{E}) \arrow[r, "p"] \arrow[d, "s"] & \mathcal{C}^{\Delta^1} \arrow[d, "s"] \\
\mathcal{E} \arrow[r, "p"] & \mathcal{C}
\end{tikzcd}$$

is a pullback square.

Remark 2.1.38. Concretely speaking, a cocartesian fibration is a functor with the property that, for any $x \in \mathcal{E}$ and any morphism $g\colon px \to c$ in $\mathcal{C}$, there is a unique p–cocartesian morphism $\tilde{g}\colon x \to \tilde{c}$ in $\mathcal{E}$ which maps to $g\colon px \to c$ under the functor p.

Construction 2.1.39 (Straightening–unstraightening)**.** For a functor $F\colon I \to \mathrm{Cat}_\infty$, there is a construction producing an ∞–category called the *unstraightening* $\mathrm{Un}(F)$ equipped with a map to I satisfying the property of being a cocartesian fibration. The point of this construction is that there is then an equivalence of ∞–categories

$$\mathrm{coCart}(I) \simeq \mathrm{Fun}(I, \mathrm{Cat}_\infty)$$

where the left hand side denotes the ∞–category of cocartesian fibrations over I. This is a difficult theorem first proved by Lurie in the ∞–categorical setting in [Lur12] with subsequent easier proofs by many other people. Suffice to say, when the functor F is constant with value $\mathcal{C} \in \mathrm{Cat}_\infty$, $\mathrm{Un}(F) \to I$ is given simply by the projection $I \times \mathcal{C} \to I$.

Among other things, we can use this construction to compute limits in Cat_∞. To formulate this, we will need the following notations:

Notation 2.1.40. For a functor $p\colon \mathcal{E} \to \mathcal{C}$, we define the *category of sections* $\Gamma(p)$ as the pullback

$$\begin{tikzcd}
\Gamma(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{C},\mathcal{E}) \arrow[d, "p"] \\
\{\mathrm{id}\} \arrow[r] & \mathrm{Fun}(\mathcal{C},\mathcal{C})
\end{tikzcd}$$

We write $\Gamma_{\mathrm{cocart}}(p) \subseteq \Gamma(p)$ for the full subcategory of those sections which take send every morphism in $\mathcal{C}$ to a p–cocartesian morphism in $\mathcal{E}$.

Theorem 2.1.41 (Lurie, [HW21, Prop I.36])**.** *Given a functor $F\colon I \to \mathrm{Cat}_\infty$, we have the following formulae for (co)limits:*

$$\operatorname*{colim}_I F \simeq \mathrm{Un}(F)[\{\textit{cocart edges}\}^{-1}] \quad \textit{and} \quad \lim_I F \simeq \Gamma_{\mathrm{cocart}}(\mathrm{Un}(F) \to I)$$

In particular, if $F\colon I \to \mathrm{An} \subseteq \mathrm{Cat}_\infty$, then we have

$$\operatorname*{colim}_I F \simeq |\mathrm{Un}(F)| \quad \textit{and} \quad \lim_I F \simeq \Gamma(\mathrm{Un}(F) \to I)$$

2.2 Presentability

A presentable ∞-category is, roughly, a cocomplete ∞-category (thus, usually a large ∞-category) which is controlled by "small" objects, and thus describable with a set's worth of data. In more detail:

Definition 2.2.1. An ∞-category $\mathcal{C}$ is presentable if it is cocomplete, and there is a cardinal κ and a small set of κ-compact objects $S \subset \mathcal{C}$ that generate $\mathcal{C}$ under colimits.

This is, in principle, a very strong restriction, but in practice, most "natural" cocomplete ∞-categories you will encounter are presentable[2], so that it almost becomes a mild assumption. Here are some other characterisations of presentability that might be useful in different situations.

Theorem 2.2.2 (Simpson, Lurie, [Lur12, Thm. 5.5.1.1])**.** *Let $\mathcal{C}$ be a category. The following are equivalent conditions:*

(a) $\mathcal{C}$ is presentable,

(b) there exists a small category $\mathcal{D}$ participating in a Bousfield localisation

$$\mathrm{PSh}(\mathcal{D}) \underset{i}{\overset{L}{\rightleftarrows}} \mathcal{C}$$

where the endofunctor $iL\colon \mathrm{PSh}(\mathcal{D}) \to \mathrm{PSh}(\mathcal{D})$ preserves κ–filtered colimits for some regular cardinal κ,

(c) there exists a regular cardinal κ and a small category $\mathcal{D}$ admitting κ–small colimits such that there is an equivalence $\mathrm{Ind}_\kappa\mathcal{D} \xrightarrow{\simeq} \mathcal{C}$.

Exercise 2.2.3 (Completeness of presentable categories)**.** Use Proposition 2.1.35 to show that in a Bousfield localisation $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : i$, the subcategory $\mathcal{D}$ is always closed under limits that exist in $\mathcal{C}$. Hence, deduce from Theorem 2.2.2 that presentable categories admit small limits.

---

[2] With the caveat that the opposite of a presentable ∞-category is almost never presentable.

This type of size control affords many pleasant properties, among which the existence of objects satisfying various universal properties. This is neatly encoded in the adjoint functor theorem:

Theorem 2.2.4 (Adjoint Functor Theorem (AFT), [Lur12, Cor. 5.5.2.9]). *Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{D}$ be a cocomplete ∞-category.*

1. *A functor $f : \mathcal{C} \to \mathcal{D}$ is a left adjoint if and only if it preserves colimits.*
2. *Suppose $\mathcal{D}$ is also presentable. In this case, a functor $g : \mathcal{D} \to \mathcal{C}$ is a right adjoint if and only if it preserves limits and is accessible.*

"Recall" that a functor is called accessible if it preserves κ-filtered colimits for some cardinal κ. As for presentability, this is in principle rather restrictive, but in practice, most functors you've encountered are accessible (it is in fact quite hard to come up with a non-accessible functor).

Remark 2.2.5. Item 2 in the AFT, specialized to $\mathcal{C} = \mathrm{An}$ is a corepresentability criterion. Indeed, for $\mathcal{D}$ cocomplete, a functor $g : \mathcal{D} \to \mathrm{An}$ is corepresentable if and only if it admits a left adjoint. Hence, by the AFT, when $\mathcal{D}$ is presentable, this is the case if and only if it is accessible and preserves limits.

Item 1, specialized to $\mathcal{D} = \mathrm{An}^{\mathrm{op}}$ is a representability criterion[3], in the same way. Thus, for presentable $\mathcal{C}$, a functor $f : \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ is representable if and only if it preserves limits.

Theorem 2.2.6 (Reflection theorem, [RS22, Thm. 1.1]). *Let $\mathcal{C}$ be a presentable category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory which is closed under limits and κ–filtered colimits for some regular cardinal κ. Then, $\mathcal{D}$ is itself presentable and in particular, by Theorem 2.2.4, the inclusion $\mathcal{D} \subseteq \mathcal{C}$ admits a left adjoint and so participates in a Bousfield localisation. presentable*

Next comes a bridge between the "small" and the "big" worlds of stable ∞–categories:

Theorem 2.2.7 ([Lur12, Prop. 5.5.7.8]). *The Ind–completion and compact objects functor participate in the following equivalence of ∞–categories*

$$\mathrm{Ind}\colon \mathrm{Cat}_\infty^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_{L,\mathrm{st},\omega} : (-)^\omega$$

where $\mathrm{Cat}_\infty^{\mathrm{perf}}$ denotes the ∞–category of small idempotent–complete stable ∞–categories and $\mathrm{Pr}_{L,\mathrm{st},\omega}$ denote the ∞–category of ω–compactly generated presentable stable ∞–categories and morphisms given by functors which preserve colimits and compact objects.

---

[3] Here, it's important that this part of the statement does not require $\mathcal{D}$ to be presentable, as $\mathrm{An}^{\mathrm{op}}$ is not presentable.

In the “big world”, that is, when $\mathcal{C}$ has all small colimits, there is a better suited notion of subcategory, namely:

Definition 2.2.8. Let $\mathcal{C}$ be a cocomplete stable ∞–category. A stable subcategory $\mathcal{D}$ of $\mathcal{C}$ is called *localizing* if it is closed under all colimits.

In this situation, we have:

Proposition 2.2.9. *Suppose $\mathcal{C}$ is a cocomplete stable ∞–category and $\mathcal{D}$ is a localizing subcategory of $\mathcal{C}$. In this case, $\mathcal{C}/\mathcal{D}$ is also cocomplete, and $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ preserves colimits.*

If $\mathcal{C}, \mathcal{D}$ are furthermore presentable, then $\mathcal{C}/\mathcal{D}$ is an accessible localization of $\mathcal{C}$, i.e. it is also presentable and $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ admits a fully faithful right adjoint with essential image the c's in $\mathcal{C}$ such that for all $d \in \mathcal{D}$, $\mathrm{Map}_{\mathcal{C}}(d, c) = 0$, or equivalently, for all $d \in \mathcal{D}, \mathrm{map}_{\mathcal{C}}(d, c) = 0$.

Observation 2.2.10 (Colimits in presentables)*.* This is a very important standard categorical manoeuvre that people often use. Lurie showed that the nonfull inclusions

$$\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}} \qquad \mathrm{Pr}^R \subset \widehat{\mathrm{Cat}}$$

creates limits, and so in principle we know how to compute limits in both Pr^L and Pr^R, since limits in $\widehat{\mathrm{Cat}}$ are “easy” since we know how to compute pullbacks and arbitrary products there.

To compute colimits in Pr^L, we may leverage the equivalence $(\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathrm{Pr}^R$ (by passing to right (resp. left) adjoints as follows: suppose we have a diagram $D\colon J \to \mathrm{Pr}^L$. Then we may consider the diagram

$$\overline{D}\colon J^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} (\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathrm{Pr}^R \subset \widehat{\mathrm{Cat}}$$

so that by

$$\overset{\mathrm{Pr}^L}{\operatorname*{colim}_{J}} D \simeq \overset{\mathrm{Pr}^R}{\lim_{J^{\mathrm{op}}}} \overline{D} \simeq \overset{\widehat{\mathrm{Cat}}}{\lim_{J^{\mathrm{op}}}} \overline{D},$$

we may compute the colimit of J in Pr^L by passing to right adjoints to get a diagram indexed over J^{op} in $\widehat{\mathrm{Cat}}$ and compute the limit there.

2.3 Semiadditivity and additivity

Construction 2.3.1. Let $\mathcal{E}$ be a pointed category with finite (co)products. We sketch the construction of a natural transformation of functors $\mathcal{E} \times \mathcal{E} \to \mathcal{C}$

$$\coprod \Rightarrow \prod$$

as follows: write $r\colon * \sqcup * \to *$ for the unique map. Then $\coprod \simeq r_!$ and $\prod \simeq r_*$, and so by adjunction, constructing the desired transformation is equivalent to constructing a transformation $\mathrm{id} \Rightarrow r^*r_*$. For $(x,y) \in \mathcal{E} \times \mathcal{E}$, this is tantamount to constructing a map

$$(x,y) \longrightarrow (x \times y, x \times y)$$

for which we can use the zero object to get natural maps

$$x \xrightarrow{(1,0)} x \times y \qquad y \xrightarrow{(0,1)} x \times y$$

yielding the desired map. This is not quite a fully honest ∞–construction since we have only constructed the maps for fixed (x,y) without checking functoriality and so on which is best handled by a standard result called smooth–proper basechange which we will not cover in this cursory recollection.

Definition 2.3.2. Let $\mathcal{E}$ be a pointed category with finite (co)products. We say that it is *semiadditive* if the canonical transformation $\coprod \Rightarrow \prod$ of functors $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$ constructed above is an equivalence.

Notation 2.3.3. Since in a semiadditive category, the finite coproducts and finite products are canonically identified, we will the notations $\oplus, \sqcup$, and $\times$ interchangeably for these finite biproducts in semiadditive categories.

Construction 2.3.4 (Shear maps)**.** Let $\mathcal{E}$ be a semiadditive category and $x \in \mathcal{E}$. We may define a canonical "multiplication" map for x as the composition

$$\mu \colon x \times x \simeq x \sqcup x \xrightarrow{\text{adjunction counit}} x$$

We define the *shear map* for x to be the map

$$x \times x \xrightarrow{\mu \times \mathrm{pr}_2} x \times x$$

where pr_2 is the projection onto the second component.

Definition 2.3.5. Let $\mathcal{E}$ be semiadditive and $x \in \mathcal{E}$. We say that x is *grouplike* if the shear map $x \times x \to x \times x$ is an equivalence. We will write $\mathcal{E}^{\mathrm{gp}} \subseteq \mathcal{E}$ for the full subcategory of grouplike objects.

Definition 2.3.6. A semiadditive category $\mathcal{E}$ is said to be *additive* if the inclusion $\mathcal{E}^{\mathrm{gp}} \subseteq \mathcal{E}$ is an equivalence, that is, all objects are grouplike.

Notation 2.3.7. Let $\mathcal{E}$ be a semiadditive category admitting pullbacks and pushouts. We write $\Omega, \Sigma \colon \mathcal{E} \to \mathcal{E}$ for the functors

$$\Omega \colon X \mapsto 0 \times_X 0 \qquad \Sigma \colon X \mapsto 0 \cup_X 0$$

respectively. In particular, it is clear that we have an adjunction $\Sigma \colon \mathcal{E} \rightleftarrows \mathcal{E} : \Omega$.

Construction 2.3.8 ($\mathbb{E}_\infty$–monoids and $\mathbb{E}_\infty$–groups)**.** Write Fin_* for the 1–category of finite pointed sets and pointed maps. We shall denote by $* \in \mathrm{Fin}_*$ the singleton pointed set and for any $n \in \mathbb{N}$, $\langle n\rangle \in \mathrm{Fin}_*$ will denote the finite pointed set $\{1, 2, \ldots, n\} \sqcup *$ of size $n+1$.

Let $\mathcal{C}$ be a category admitting finite products (and in particular, the final object). We define *the category of* $\mathbb{E}_\infty$*–monoids and of* $\mathbb{E}_\infty$*–groups in* $\mathcal{C}$, respectively, as

$$\mathrm{CMon}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\mathrm{seg}}(\mathrm{Fin}_*, \mathcal{C}) \qquad\qquad \mathrm{CGrp}(\mathcal{C}) \coloneqq \mathrm{CMon}(\mathcal{C})^{\mathrm{gp}}$$

where $\mathrm{Fun}^{\mathrm{seg}}(\mathrm{Fin}_*, \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ is the full subcategory on functors $F\colon \mathrm{Fin}_* \to \mathcal{C}$ satisfying the *Segal condition*, i.e. $F(*) = *_{\mathcal{C}}$ and for all $n \in \mathbb{N}$ and $i \leq n$, the maps $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ given by sending $i \mapsto 1$ and $j \mapsto *$ for $j \neq i$, the induced map in $\mathcal{C}$

$$\prod_i F(\rho_i)\colon F(\langle n\rangle) \longrightarrow \prod_i F(\langle 1\rangle)$$

is an equivalence.

Exercise 2.3.9 (Intuition for $\mathbb{E}_\infty$–monoids)**.** Another interesting map in Fin_* is $\nabla\langle 2\rangle \to \langle 1\rangle$ given by sending both 1 and 2 in $\langle 2\rangle$ to 1 in $\langle 1\rangle$. Using this, we obtain from the structure of $M \in \mathrm{CMon}(\mathcal{C})$ a “multiplication map” on $M(\langle 1\rangle) \in \mathcal{C}$ defined as the following composition

$$\mu\colon M(\langle 1\rangle) \times M(\langle 1\rangle) \xleftarrow[\rho_1\times\rho_2]{\simeq} M(\langle 2\rangle) \xrightarrow{M(\nabla)} M(\langle 1\rangle)$$

Identify the commuting triangle in Fin_* which encodes that this multiplication map is “commutative”, and try to work out other diagrams in Fin_* which encode “associativity” and “unitality” of the multiplication.

Notation 2.3.10. We write $\mathrm{Cat}^{\prod} \subset \mathrm{Cat}$ for the non–full subcategory of small categories with finite products and finite product preserving functors, and we write $\mathrm{Cat}^{\mathrm{add}} \subseteq \mathrm{Cat}^{\mathrm{sadd}} \subseteq \mathrm{Cat}^{\prod}$ for the full subcategories of additive and semiadditive categories, respectively.

Theorem 2.3.11 (Omnibus semiadditivisation and additivisation)**.**

1. *The inclusions* $\mathrm{Cat}^{\mathrm{sadd}} \subseteq \mathrm{Cat}^{\prod}$ *and* $\mathrm{Cat}^{\mathrm{add}} \subseteq \mathrm{Cat}^{\prod}$ *participate in the Bousfield colocalisations*

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{sadd}} \arrow[r, hook, shift left] & \mathrm{Cat}^{\prod} \arrow[l, shift left, "\mathrm{CMon}"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{Cat}^{\mathrm{add}} \arrow[r, hook, shift left] & \mathrm{Cat}^{\prod} \arrow[l, shift left, "\mathrm{CGrp}"] \end{tikzcd}$$

   where the adjunction counits $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ *are given by restriction along the inclusion* $* \hookrightarrow \mathrm{Fin}_*$ *of the object* $\langle 1\rangle \in \mathrm{Fin}_*$*,*

2. *The adjunction counits* $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ *and* $\mathrm{CGrp}(\mathcal{C}) \to \mathcal{C}$ *are conservative,*

3. *In fact, this is also in some sense an* $(\infty, 2)$*–adjunction even for large categories. More precisely, for any* $\mathcal{E} \in \mathrm{Cat}^{\mathrm{sadd}}$ *and* $\mathcal{C}$ *a category (which might be large) admitting finite products, the adjunction counit induces an equivalence of categories* $\mathrm{Fun}^{\Pi}(\mathcal{E}, \mathrm{CMon}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}^{\Pi}(\mathcal{E}, \mathcal{C})$*, and similarly in the additive case.*

Proof. A proof of the Bousfield colocalisations may be found for example in [HW21, Thm. II.19]. Conservativity is an easy consequence of the Segal conditions. ☐

Remark 2.3.12. In particular, the Bousfield colocalisations in Theorem 2.3.11 says that, for example, for $\mathcal{E} \in \mathrm{Cat}^{\mathrm{sadd}}$, the adjunction counit $\mathrm{CMon}(\mathcal{E}) \to \mathcal{E}$ is an equivalence.

Corollary 2.3.13 (Canonical (semi)additive enrichment)**.** *Let* $\mathcal{C}$ *be a semiadditive category and* $\mathcal{A}$ *an additive category. Then we have canonical lifts of the mapping anima functors*

$$\begin{tikzcd} & \mathrm{CMon}(\mathrm{An}) \arrow[d, "\mathrm{fgt}"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, dashed] \arrow[r, "\mathrm{Map}_{\mathcal{C}}(-,-)"'] & \mathrm{An} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{CGrp}(\mathrm{An}) \arrow[d, "\mathrm{fgt}"] \\ \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[ur, dashed] \arrow[r, "\mathrm{Map}_{\mathcal{A}}(-,-)"'] & \mathrm{An} \end{tikzcd}$$

using the commutative monoid/group structure of objects in the first variable (and a similar lifting also holds using the structure in the second variable). This says that mapping anima in semiadditive (resp. additive) categories are canonically refined with the structure of $\mathbb{E}_\infty$*–monoids (resp.* $\mathbb{E}_\infty$*–groups).*

Proof. We only do the semiadditive case. For this, note that currying $\mathrm{Map}_{\mathcal{C}}(-,-)$ gives the Yoneda embedding

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{An})$$

Since limits in functor categories are computed in the target, this map is finite product preserving, and so by Theorem 2.3.11 this map lifts to a map

$$\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{CMon}\,\mathrm{Fun}(\mathcal{C}, \mathrm{An}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{CMon}(\mathrm{An}))$$

where the commutation of CMon with Fun is yet again because limits in functor categories are computed in the target. ☐

Remark 2.3.14. See for example [HW21, Cor. II.17] to see an argument as to why the liftings in Corollary 2.3.13 via the first and the second variables are equivalent.

Exercise 2.3.15. Using that an ordinary classical commutative monoid is an abelian group if and only if the sheaf map is an isomorphism, show that $M \in \mathrm{CMon}(\mathrm{An})$ is in $\mathrm{CGrp}(\mathrm{An})$ if and only if $\pi_0 M$ is a group. **Hint:** *equivalences in anima may be tested by checking* π_**–isomorphism.* Consequently, observe that there is a canonical lift of the loop functor on pointed anima as follows:

$$\begin{tikzcd}
\mathrm{CMon}(\mathrm{An}) \arrow[r, "\Omega"] \arrow[dr, dashed] & \mathrm{CMon}(\mathrm{An}) \\
& \mathrm{CGrp}(\mathrm{An}) \arrow[u, "\mathrm{fgt}"']
\end{tikzcd}$$

In other words, for any $x \in \mathrm{An}_*$, the object $\Omega x \in \mathrm{An}_*$ attains a canonical grouplike $\mathbb{E}_\infty$–monoid structure.

Lemma 2.3.16 (Group–completion formula)**.** *Let $\mathcal{E}$ be a semiadditive category admitting pullbacks and pushouts.*

1. *For any object $x \in \mathcal{E}$, we have that both Ωx and Σx are grouplike,*
2. *If the restriction of the suspension $\Sigma\colon \mathcal{E}^{\mathrm{gp}} \to \mathcal{E}^{\mathrm{gp}}$ (which makes sense by (1)) is fully faithful, then endofunctor $\Omega\Sigma\colon \mathcal{E} \to \mathcal{E}$ lifts to a Bousfield localisation*
$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left, "\Omega\Sigma"] & \mathcal{E}^{\mathrm{gp}} \arrow[l, shift left, hook']
\end{tikzcd}$$
As such, we will also write $(-)^{\mathrm{gp}} := \Omega\Sigma$ for the group–completion *functor.*

Proof. For part (1), we prove the case of Σx, since the other is similar. Recall that being grouplike is defined as the property of satisfying that the shear map $\mu \times \mathrm{pr}_2\colon \Sigma x \times \Sigma x \to \Sigma x \times \Sigma x$ is an equivalence. By Yoneda, this condition may be checked by checking it upon applying $\mathrm{Map}_{\mathcal{E}}(-, y)$ for all $y \in \mathcal{E}$. But then we have the identifications

$$\mathrm{Map}_{\mathcal{E}}(\Sigma x \times \Sigma x, y) \simeq \mathrm{Map}_{\mathcal{E}}(\Sigma x \sqcup \Sigma x, y) \simeq \Omega \mathrm{Map}_{\mathcal{E}}(x, y) \times \Omega \mathrm{Map}_{\mathcal{E}}(x, y),$$

and unwinding Corollary 2.3.13 yields a commuting square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{E}}(\Sigma x \times \Sigma x, y) \arrow[r, "(\mu\times\mathrm{pr}_2)^*"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathcal{E}}(\Sigma x \times Bx, y) \arrow[d, "\simeq"] \\
\Omega \mathrm{Map}_{\mathcal{E}}(x, y) \times \Omega \mathrm{Map}_{\mathcal{E}}(x, y) \arrow[r, "\mu\times\mathrm{pr}_2"] & \Omega \mathrm{Map}_{\mathcal{E}}(x, y) \times \Omega \mathrm{Map}_{\mathcal{E}}(x, y)
\end{tikzcd}$$

But by Exercise 2.3.15, the bottom horizontal map is an equivalence. This completes the proof of (1).

For part (2), we just need to show that for all $x \in \mathcal{E}$, the two maps

$$\eta_{\Omega\Sigma x}, \Omega\Sigma\eta_x \colon \Omega\Sigma x \longrightarrow \Omega\Sigma\Omega\Sigma x$$

are equivalences. Since these two maps admit a common retraction $\Omega\varepsilon_\Sigma$ by the triangle identities, it suffices to argue that $\eta_{\Omega\Sigma x}$ is an equivalence. This in turn is implied by our hypothesis that Σ was fully faithful, since this says that the restricted adjunction $\Sigma\colon \mathcal{E}^{\mathrm{gp}} \rightleftharpoons \mathcal{E}^{\mathrm{gp}} : \Omega$ is a Bousfield colocalisation, and so the adjunction unit $\eta_{\Omega\Sigma x}$ is indeed an equivalence by Observation 2.1.14, as required. □

2.4 Stability

The condition of stability on a category has been well–recognised for a long time to be a crucial idea in carrying out homological methods in generalised settings. As will perhaps become clearer in the course of these lectures, stable categories may be said to be the “universal location” in which to carry out Mayer–Vietories/excision arguments. Traditionally, its manifestations include the notion of chain complexes, triangulated categories, and spectra. One of the many key advantages of ∞–category theory over 1–categories is that this condition is most naturally a higher categorical notion, owing to the fact that the shift functors Ω and Σ (or $[1]$ and $[-1]$ for chain complexes) necessitate the notion of mapping *spaces* as opposed to just mapping sets.

In this subsection, we introduce the basics of the theory of stable ∞–categories and the various key constructions we need for the main body of these notes.

Notation 2.4.1. Let $\mathcal{C}$ be a pointed ∞–category, ie. it has a zero object. Let $X \in \mathcal{C}$. When they exist, we write ΩX (resp. ΣX) for the pullback (resp. pushout)

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] & & X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X & & 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Definition 2.4.2 ([Lur17, Def. 1.1.1.9, Prop. 1.4.2.27])**.** Let $\mathcal{C}$ be a pointed ∞–category. We say that it is *stable* if the following equivalent conditions are satisfied:

1. pullbacks and pushouts exist, and a commuting square is a pullback if and only if it is a pushout;
2. finite colimits exist, and the functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence;
3. finite limits exist, and the functor $\Omega\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.

Fact 2.4.3. It is a non–immediate fact that a stable ∞-category is in particular additive.

Remark 2.4.4 (Commuting squares, pullbacks, pushouts). Let us make precise what it means for squares to commute and what it means for them to be pullbacks/pushouts. Unlike in 1–category theory where a commuting square is just *property* for two different compositions of morphisms to agree, in higher category theory, commuting squares are extra *structures*, that is, when we write a commuting square as in the left square in

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "q"] & & A \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\equiv_\sigma"] & B \arrow[d, "q"] \\
C \arrow[r, "g"'] & D & & C \arrow[r, "g"'] & D
\end{tikzcd}$$

we are implicitly including a choice of a *witnessing homotopy* σ as made explicit in the square on the right. Thus, when we say a “commuting square”, we do not just mean the data of morphisms f, g, p, q, but also the datum σ of an equivalence $\sigma\colon q\circ f \simeq g\circ p$ between the two compositions.

In light of this, when we speak of pullback/squares, we mean that the data (f, g, p, q, σ) has the property of being a pullback/pushout. It could very well happen that another commuting data (f, g, p, q, τ) for a different choice of homotopy $\tau\colon q\circ f \simeq g\circ p$ may no longer be a pullback even though the collection of morphisms are still the same. Unwinding, for instance, the universal property of being a pullback, which is articulated succinctly as the induced map on mapping anima

$$\mathrm{Map}_{\mathcal{C}}(X, A) \xrightarrow[\simeq]{f\times_{(qf\simeq gp)}p} \mathrm{Map}_{\mathcal{C}}(X, B)\times_{\mathrm{Map}_{\mathcal{C}}(X,D)}\mathrm{Map}_{\mathcal{C}}(X, C) \tag{2.1}$$

being an equivalence for $X\in\mathcal{C}$, this says concretely speaking that given a commuting square data (k, l, q, g, τ) on the left in

$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "l"'] \arrow[dr, phantom, "\equiv_\tau"] & B \arrow[d, "q"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "l"'] & & \\
& A \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\equiv_\sigma"] & B \arrow[d, "q"] \\
& C \arrow[r, "g"'] & D
\end{tikzcd}$$

with $\equiv_\beta$ between φ and k, and $\equiv_\gamma$ between φ and l,

there is a contractible space (in particular, it is also nonempty!) of tuples

$$\bigl(\varphi,\ \beta\colon f\circ\varphi\simeq k,\ \gamma\colon p\circ\varphi\simeq l,\ \omega\colon\langle\sigma,\beta,\gamma\rangle\simeq\tau\bigr)$$

where $\langle\sigma,\beta,\gamma\rangle$ is the commuting datum for the outer square in the diagram on the right obtained by gluing the homotopies σ,β, and γ, and ω is a homotopy witnessing that the outer square on the right is equivalent to the square on the left.

As is perhaps clear from these explanations, the *meanings* of the symbols in higher category theory can be off-puttingly complicated and nuanced, but one of the aims of the course is to convince you that when manipulated judiciously (as for example by expressing and manipulating the universal property of pullbacks given by (2.1)), one can get quite far in life (though of course not always, especially in specific computational situations!) without being bogged down by all these homotopies. Put simply, handled correctly (read: like an ∞–engineer), the *syntax* of higher categories is often as difficult/easy as 1–categories, even though its *semantics* may be hard/rich/subtle.

Terminology 2.4.5 ((Co)fibre sequences)**.** Let $\mathcal{C}$ be a pointed category. A fibre sequence (resp. cofibre sequence), which we will often cavalierly write as the data

$$X\xrightarrow{f}Y\xrightarrow{g}Z$$

of morphisms in $\mathcal{C}$, is more precisely the data of a commuting square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\equiv"] & Y \arrow[d, "g"] \\
0 \arrow[r] & Z
\end{tikzcd}$$

which has the property of being a pullback square (resp. pushout square). In such cases, we will write $X = \mathrm{fib}(g)$ (resp. $Z = \mathrm{cofib}(f)$). To emphasise that in fibre sequences are equivalent to cofibre sequences, we will sometimes say *bifibre sequence* to mean a fibre sequences.

Exercise 2.4.6 ("5 A Day")**.** The following exercises are very instructive in the way of getting used to manipulating bifibre sequences in stable categories.

1. (Self–duality): Show that $\mathcal{C}$ is stable if and only if $\mathcal{C}^{\mathrm{op}}$ is stable,

2. (Morphisms-to-objects principle): Show that a morphism $f\colon x \to y$ is an equivalence if and only if $\mathrm{fib}(f) \simeq 0$ if and only if $\mathrm{cofib}(f) \simeq 0$,

3. (Squares–to–morphisms principle): Show that a commuting square

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "p"'] & y \arrow[d, "q"] \\
w \arrow[r, "g"'] & z
\end{tikzcd}$$

   is a pullback (and thus equivalently a pushout) if and only if any one of the induced maps $\mathrm{fib}(f) \to \mathrm{fib}(g), \mathrm{fib}(p) \to \mathrm{fib}(q), \mathrm{cofib}(f) \to \mathrm{cofib}(g)$, and $\mathrm{cofib}(p) \to \mathrm{cofib}(q)$ is an equivalence. **Hint:** *show for example that* $\mathrm{fib}(x \to w \times_z y) \simeq \mathrm{fib}\big(\mathrm{fib}(f) \to \mathrm{fib}(g)\big)$,

4. (Bifibre rotations): Show that for every bifibre sequence $x \to y \to z$, there are canonical bifibre sequences $\Omega z \to x \to y$ and $y \to z \to \Sigma x$,

5. (Fibres of compositions): Let $f\colon x \to y$ and $g\colon y \to z$ be maps in a stable category. Compute $\mathrm{fib}(g \circ f)$ in terms of $\mathrm{fib}(f)$ and $\mathrm{fib}(g)$.

Proposition 2.4.7. *A functor between stable ∞–categories preserves finite colimits if and only if it preserves finite limits.*

Notation 2.4.8. Let $\varphi\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories. We say that it is left exact (resp. right exact) if it preserves finite limits (resp. finite colimits). We write $\mathrm{Cat}^{\mathrm{lex}}, \mathrm{Cat}^{\mathrm{rex}} \subset \mathrm{Cat}$ for the non–full subcategories of Cat consisting of small categories admitting finite limits (resp. colimits) and left exact (resp. right exact) functors. Since a functor of stable categories preserve finite limits if and only if it preserve finite colimits, we will say that it is an *exact functor* if it satisfies either one (and so both) of the equivalent conditions. We shall write $\mathrm{Cat}^{\mathrm{ex}}$ for the full subcategory of either $\mathrm{Cat}^{\mathrm{rex}}$ or $\mathrm{Cat}^{\mathrm{lex}}$ consisting of the stable categories and exact functors between them.

Fact 2.4.9. The inclusion $\mathrm{Cat}^{\mathrm{lex}} \subset \mathrm{Cat}$ creates limits, i.e. limits of diagrams in $\mathrm{Cat}^{\mathrm{lex}}$ are computed in Cat.

Construction 2.4.10 (Stabilisations, [Lur17, Cor. 1.4.2.23])**.** Let $\mathcal{D} \in \mathrm{Cat}^{\mathrm{lex}}$ and write $\mathcal{D}_* \coloneqq \mathcal{D}_{/*}$ for the category of pointed objects. It is a straightforward check that this is now a pointed category. We then define the *category of spectral objects in* $\mathcal{D}$ as

$$\mathrm{Sp}(\mathcal{D}) \coloneqq \lim(\cdots \xrightarrow{\Omega} \mathcal{D}_* \xrightarrow{\Omega} \mathcal{D}_*)$$

Note by virtue of Fact 2.4.9, this limit may be taken in Cat. There is a canonical functor $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ given by the composition

$$\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \xrightarrow{\text{projection to the lowest term}} \mathcal{D}_* \xrightarrow{\text{fgt}} \mathcal{D}$$

As for Theorem 2.3.11 and Corollary 2.3.13, we have the following pair of results for stabilisations:

Theorem 2.4.11 (Omnibus stabilisation)**.**

1. *The inclusion* $\mathrm{Cat}^{\mathrm{ex}} \subseteq \mathrm{Cat}^{\mathrm{lex}}$ *participates in the Bousfield colocalisation*

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{ex}} \arrow[r, hook, shift left] & \mathrm{Cat}^{\mathrm{lex}} \arrow[l, shift left, "\mathrm{Sp}"] \end{tikzcd}$$

   whose adjunction counit is given by the map $\Omega^\infty \colon \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$*,*

2. *In fact, this is also in some sense an* $(\infty, 2)$*–adjunction even for large categories. More precisely, for any* $\mathcal{E} \in \mathrm{Cat}^{\mathrm{ex}}$ *and* $\mathcal{C}$ *a category (which might be large) admitting finite limits, the adjunction counit induces an equivalence of categories* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}, \mathrm{Sp}(\mathcal{C})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{E}, \mathcal{C})$*.*

Corollary 2.4.12 (Canonical spectral enrichment)**.** *Let* $\mathcal{A}$ *be a stable category. Then we have canonical lifts of the mapping anima functor*

$$\begin{tikzcd} & \mathrm{Sp}(\mathrm{An}) \arrow[d, "\Omega^\infty"] \\ \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[ur, dashed] \arrow[r, "\mathrm{Map}_{\mathcal{A}}(-,-)"'] & \mathrm{An} \end{tikzcd}$$

using the stability in the first variable (and a similar lifting also holds using the stability in the second variable). This says that mapping anima in stable categories are canonically refined with the structure of a spectrum.

Definition 2.4.13. A stable subcategory of a stable ∞–category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ which is stable under finite limits and colimits.

It is called *thick* if it is furthermore closed under retracts in $\mathcal{C}$ (and a "thick subcategory" is implicitly assumed to be a stable subcategory). If $\mathcal{C}$ is idempotent-complete, this latter condition is equivalent to $\mathcal{D}$ being idempotent-complete.

One of the reasons to consider thick subcategories is that they are precisely the kernels of exact functors. One direction (that kernels of exact functors are thick subcategories) is an easy observation, and the other direction comes from Verdier quotients, which we describe in the following omnibus theorem (good sources for which include [NS18, Thm. I.3.3] and [CDH+20, § A.2]):

Theorem 2.4.14 (Omnibus Verdier quotients)**.** *Given a thick subcategory $\mathcal{D}$ of a stable ∞–category $\mathcal{C}$, there is a stable ∞–category $\mathcal{C}/\mathcal{D}$, called the Verdier quotient of $\mathcal{C}$ by $\mathcal{D}$, with a projection functor $p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ and the following properties:*

1. *$\mathcal{C}/\mathcal{D}$ is the cofiber of the inclusion $\mathcal{D} \to \mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ and as such induces, for every stable ∞–category $\mathcal{E}$, a fully faithful functor $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}/\mathcal{D}, \mathcal{E}) \xrightarrow{p^*} \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ with essential image those exact functors $\mathcal{C} \to \mathcal{E}$ that vanish on $\mathcal{D}$.*
2. *The kernel of p is exactly $\mathcal{D}$.*
3. *$p : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ witnesses the latter as the localization of $\mathcal{C}$ at "mod-$\mathcal{D}$ equivalences", i.e. maps $f : x \to y$ whose cofiber (or equivalently, fiber) is in $\mathcal{D}$. In particular p is essentially surjective.*
4. *For any pair of objects $x, y \in \mathcal{C}$, the canonical map $\mathrm{colim}_{f\in\mathcal{D}_{/y}} \mathrm{Map}_{\mathcal{C}}(x, \mathrm{cofib}(f)) \to \mathrm{Map}_{\mathcal{C}/\mathcal{D}}(p(x), p(y))$ is an equivalence, and $\mathcal{D}_{/y}$ is a filtered ∞-category. Dually, the canonical map $\mathrm{colim}_{g\in\mathcal{D}_{x/}} \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(g), y) \to \mathrm{Map}_{\mathcal{C}/\mathcal{D}}(p(x), p(y))$ is an equivalence.*

Finally, if $\mathcal{D}$ is only a stable subcategory, then the theorem remains true except for Item 2 which is replaced by "the kernel of p is exactly the closure of $\mathcal{D}$ under retracts in $\mathcal{C}$".

Exercise 2.4.15.

1. Work out exactly what "the canonical map" is in Item 4.
2. Deduce that the formula for mapping spaces in Item 4 also holds for mapping spectra.

2.5 General multiplicative matters

Symmetric monoidal structures in the ∞–categorical setting are much more intricate than their 1–categorical counterparts, the reason being that we have to specify a lot more coherence structures. While this can be done very neatly by a key insight of Graeme Segal in [Seg74], we will nevertheless forego most of the precise discussions of these matters in the interest of brevity. As such, we will content ourselves mostly with informal "definitions" in this subsection, and refer the reader to [Lur17, §2.1] for more details on the basic precise notions. In particular, we will not cover the notion of *operads* in these notes.

Definition 2.5.1. A symmetric monoidal category (resp. symmetric monoidal functor) is an object (resp. morphism) in CMon(Cat).

Idea 2.5.2. A *symmetric monoidal ∞–category* $\mathcal{C}^{\otimes}$ is roughly speaking an ∞–category equipped with symmetric monoidal structures which include a tensor map

$$- \otimes - \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

which is coherently associative and commutative, as well as a unit object $\mathbb{1} \in \mathcal{C}$ such that $\mathbb{1} \otimes (-) \simeq \mathrm{id}_{\mathcal{C}}$. Given this, we will often also denote $\mathcal{C}^{\otimes}$ with $(\mathcal{C}, \otimes, \mathbb{1})$.

A symmetric monoidal functor $f^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ between symmetric monoidal ∞–categories is an underlying functor $f \colon \mathcal{C} \to \mathcal{D}$ together with structures witnessing the compatibility of f with $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$. For example, it includes the datum of an equivalence

$$\mathbb{1}_{\mathcal{D}} \xrightarrow{\simeq} f(\mathbb{1}_{\mathcal{C}})$$

and it will also include the data of equivalences

$$f(X) \otimes_{\mathcal{D}} f(Y) \xrightarrow{\simeq} f(X \otimes_{\mathcal{C}} Y)$$

for every $X, Y \in \mathcal{C}$ etc.

“Definition” 2.5.3. A *lax symmetric monoidal functor* $f^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ has the same structures as a symmetric monoidal functor, except that the maps

$$\mathbb{1}_{\mathcal{D}} \to f(\mathbb{1}_{\mathcal{C}}) \qquad f(X) \otimes_{\mathcal{D}} f(Y) \to f(X \otimes_{\mathcal{C}} Y)$$

are no longer required to be equivalences.

Warning 2.5.4. Unlike in 1–categories, a (lax) symmetric monoidal functor is *not* the same as a (lax) monoidal functor satisfying some properties in the higher setting! The “symmetric” in a “(lax) symmetric monoidal functor” $F \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ encodes, for example, also the data of commutations

$$\begin{tikzcd}
F(x) \otimes_{\mathcal{D}} F(y) \arrow[r, "\mathrm{lax}"] \arrow[d, "\tau_{Fx,Fy}"', "\simeq"] \arrow[dr, phantom, "\equiv"] & F(x \otimes_{\mathcal{C}} y) \arrow[d, "F\tau_{x,y}"', "\simeq"] \\
F(y) \otimes_{\mathcal{D}} F(x) \arrow[r, "\mathrm{lax}"] & F(y \otimes_{\mathcal{C}} x)
\end{tikzcd}$$

The point is that in 1–categories, it is just a *property* for this square to commute, whereas in higher categories, it is a *structure* that witnesses that this square commutes.

“Definition” 2.5.5. In a symmetric monoidal ∞–category $\mathcal{C}^{\otimes}$, one may speak of an $\mathbf{E}_{\infty}$–algebra object (we will often just call these commutative algebra objects). Writing $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ for the ∞–category of commutative algebra objects, roughly speaking with some justifiable abuse of notations, an object $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$ is the datum of an object $A \in \mathcal{C}$ equipped with “multiplication maps”

$$\mu \colon A \otimes_{\mathcal{C}} A \longrightarrow A$$

which is coherently associative and commutative, as well as a "unit map" $1\colon \mathbb{1} \to A$ together with the datum of a homotopy to the identity id_A for the composite

$$A \simeq A \otimes \mathbb{1} \xrightarrow{\mathrm{id}\otimes 1} A \otimes A \xrightarrow{\mu} A$$

The category $\mathrm{CAlg}(\mathcal{C}^{\otimes})$ comes equipped with a natural forgetful functor $\mathrm{fgt}\colon \mathrm{CAlg}(\mathcal{C}^{\otimes}) \to \mathcal{C}$ which just remembers the underlying object A.

Fact 2.5.6. Lax symmetric monoidal functors preserve commutative algebra objects, ie. any lax symmetric monoidal functor $f^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ induces a functor

$$f\colon \mathrm{CAlg}(\mathcal{C}^{\otimes}) \longrightarrow \mathrm{CAlg}(\mathcal{D}^{\otimes})$$

Morally, this is because if we started with an $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$, then the lax maps provide us with enough structure to define a multiplication map

$$f(A) \otimes_{\mathcal{D}} f(A) \xrightarrow{\mathrm{lax}} f(A \otimes_{\mathcal{C}} A) \xrightarrow{f\mu} f(A)$$

and so on.

Note that it is very important that we specify which symmetric monoidal structure we are using when extracting the category of $\mathbb{E}_\infty$–algebras. In particular, the symbol $\mathrm{CAlg}(\mathcal{C})$ technically does not make sense and we must write $\mathrm{CAlg}(\mathcal{C}^{\otimes})$, although we will often abuse this and write something like $\mathrm{CAlg}(\mathrm{Sp})$ since the symmetric monoidal structure will always be understood. The following proposition and the ensuing example should drive home this point.

Proposition 2.5.7. *Let $\mathcal{C}$ be a category admitting finite products and $\mathcal{C}^{\times}$ the cartesian symmetric monoidal structure. Then there is an equivalence* $\mathrm{CAlg}(\mathcal{C}^{\times}) \simeq \mathrm{CMon}(\mathcal{C})$*.*

Example 2.5.8. We have equivalences of categories

$$\mathrm{CAlg}(\mathrm{Ab}^{\times}) \simeq \mathrm{CMon}(\mathrm{Ab}) \simeq \mathrm{Ab} \qquad\qquad \mathrm{CAlg}(\mathrm{Ab}^{\otimes}) \simeq \mathrm{CRing}$$

where we have used Theorem 2.3.11 and that the category of abelian groups Ab is already semiadditive.

The following pair of results are bread–and-butter "coherence machines" that allow us to build new symmetric monoidal categories and functors from old ones. A convenient source for a very general discussion of the first result is [HHL+21].

Proposition 2.5.9. *Let $L^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor whose underlying functor $L\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$. In this case, the right adjoint can canonically be refined with the structure of a lax symmetric monoidal functor $R^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$. Moreover, these adjunctions participate in a square of adjunctions*

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}^{\otimes}) \arrow[r, shift left, "L"] \arrow[d, "\mathrm{fgt}"'] & \mathrm{CAlg}(\mathcal{D}^{\otimes}) \arrow[l, shift left, "R"] \arrow[d, "\mathrm{fgt}"] \\
\mathcal{C} \arrow[r, shift left, "L"] & \mathcal{D} \arrow[l, shift left, "R"]
\end{tikzcd}$$

that is, the adjunction (co)units of the top adjunction are preserved by the forgetful functors.

Proposition 2.5.10 ([Lur17, Prop. 2.2.1.9])**.** *Suppose $\mathcal{C}^{\otimes} \in \mathrm{CMon}(\mathrm{Cat})$ and $L\colon \mathcal{C} \to \mathcal{D}$ a Bousfield localisation. Suppose furthermore that for every morphism $x \to y$ in $\mathcal{C}$ that gets sent to an equivalence in $\mathcal{D}$ under L, the morphisms $x \otimes z \to y \otimes z$ also get sent to equivalences in $\mathcal{D}$ under L for all $z \in \mathcal{C}$. Then there exists a symmetric monoidal structure $\mathcal{D}^{\otimes}$ on $\mathcal{D}$ and an enhancement of the functor L to a symmetric monoidal one.*

Fact 2.5.11 (The Lurie tensor product and presentably symmetric monoidal ∞–categories)*.* There is a symmetric monoidal structure on Pr_L, which is usually called the Lurie tensor product, whose tensor unit is the ∞–category of spaces An. This tensor product is defined by the following universal property: writing $\mathrm{Fun}^{L,L}(\mathcal{C}\times\mathcal{D},\mathcal{E})$ for the full subcategory of bicocontinuous functors (ie. those which preserve small colimits in each variable), the Lurie tensor product $\mathcal{C}\otimes\mathcal{D}$ is equipped with a bicocontinuous functor $\mathcal{C}\times\mathcal{D}\to\mathcal{C}\otimes\mathcal{D}$ which induces an equivalence

$$\mathrm{Fun}^{L}(\mathcal{C}\otimes\mathcal{D},\mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{L,L}(\mathcal{C}\times\mathcal{D},\mathcal{E})$$

Under this symmetric monoidal structure, the non–full inclusion $\mathrm{Pr}_L \hookrightarrow \widehat{\mathrm{Cat}}$ naturally enhances to a lax symmetric monoidal functor $\mathrm{Pr}_L^{\otimes} \hookrightarrow \widehat{\mathrm{Cat}}^{\times}$.

As a consequence of the universal property of the Lurie tensor product, a commutative algebra object $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Pr}_L^{\otimes})$ is a presentable ∞–category equipped with a tensor product

$$-\otimes- \colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}\otimes\mathcal{C} \xrightarrow{\mu} \mathcal{C}$$

which is bicocontinuous. To distinguish this important extra criterion, we will call an object in $\mathrm{CAlg}(\mathrm{Pr}_L^{\otimes})$ *presentably symmetric monoidal ∞–categories.* Crucially, by the adjoint functor theorem, presentably symmetric monoidal ∞–categories are always closed symmetric monoidal in that for any $X \in \mathcal{C}$, $-\otimes X$ has a right adjoint that we denote by $\mathrm{Hom}_{\mathcal{C}}(X,-)$ called the *internal hom object.* Furthermore, the symmetric monoidal structure $\mathrm{Pr}_L^{\otimes}$ is itself a closed one since the right adjoint of $-\otimes\mathcal{C}$ is given by $\mathrm{Fun}^{L}(\mathcal{C},-)$.

To end this subsection on multiplicative structures, we will explain one blockbuster application of the magical Lurie tensor product, namely the long elusive model–independent construction of the symmetric monoidal structure on spectra. For this, we shall need to recall the technology of idempotent objects.

Terminology 2.5.12. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal category. An *idempotent object* is a morphism $\mathbb{1}_{\mathcal{C}} \xrightarrow{f} I$ in $\mathcal{C}$ with the property that the two morphisms

$$I \xrightarrow{f\otimes\mathrm{id}} I\otimes I \qquad\qquad I \xrightarrow{\mathrm{id}\otimes f} I\otimes I$$

are equivalences. An algebra $A \in \mathrm{CAlg}(\mathcal{C}^{\otimes})$ is said to be *idempotent* if the unit map $\mathbb{1} \xrightarrow{\eta} A$ is an idempotent object. We write $\mathrm{IdemObj}(\mathcal{C}^{\otimes})$ and $\mathrm{IdemAlg}(\mathcal{C}^{\otimes})$ for the categories of idempotent objects and idempotent algebras in $\mathcal{C}^{\otimes}$, respectively.

The following is the key "coherence machine" in this setting in that it allows us to very easily construct the unique $\mathbb{E}_\infty$–algebra structure (which seems like *a lot* to construct) in the special setting of idempotent objects.

Proposition 2.5.13 (Idempotent algebra machine and localisation)**.** *The forgetful functor* $\mathrm{IdemAlg}(\mathcal{C}^{\otimes}) \to \mathrm{IdemObj}(\mathcal{C}^{\otimes})$ *is an equivalence. Furthermore, for an idempotent algebra A, the basechange functor* $-\otimes A\colon \mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ *naturally upgrades to a symmetric monoidal Bousfield localisation* $-\otimes A\colon \mathcal{C}^{\otimes} \to \mathrm{Mod}_A(\mathcal{C})^{\otimes}$.

The point here is that it is often quite easy to check that an object (equipped with a morphism from the tensor unit) is an idempotent object, and is certainly much easier than to construct $\mathbb{E}_\infty$–algebras in general.

For the final theorem of this subsection, we will also need the following notations:

Notation 2.5.14. Write $\mathrm{Pr}_{L,*}, \mathrm{Pr}_{L,\mathrm{sadd}}, \mathrm{Pr}_{L,\mathrm{add}}, \mathrm{Pr}_{L,\mathrm{st}}$ be the full subcategories of Pr_L consisting of the pointed, semiadditive, additive, and stable categories respectively.

The following key theorem was proved by Lurie for the pointed and stable case, and by [GGN15] for the (semi)additive cases.

Theorem 2.5.15 (Omnibus modes)**.**

(a) *For all $\mathcal{C} \in \mathrm{Pr}_L$, we have natural equivalences in* Pr_L

$$\mathcal{C}\otimes\mathrm{An}_* \simeq \mathcal{C}_*, \quad \mathcal{C}\otimes\mathrm{CMon} \simeq \mathrm{CMon}(\mathcal{C}), \quad \mathcal{C}\otimes\mathrm{CGrp} \simeq \mathrm{CGrp}(\mathcal{C}), \quad \mathcal{C}\otimes\mathrm{Sp} \simeq \mathrm{Sp}(\mathcal{C})$$

(b) $\mathrm{An}_*, \mathrm{CMon}, \mathrm{CGrp}, \mathrm{Sp}$ *are all idempotent objects in* Pr_L*. In particular, there exist canonical presentably symmetric monoidal structures on these categories inducing symmetric monoidal Bousfield localisations*

$$\begin{tikzcd}
\mathrm{Pr}_L \arrow[r, shift left, "-\otimes\mathrm{An}_*"] & \mathrm{Pr}_{L,*} \arrow[l, shift left, hook'] \arrow[r, shift left, "-\otimes\mathrm{CMon}"] & \mathrm{Pr}_{L,\mathrm{sadd}} \arrow[l, shift left, hook'] \arrow[r, shift left, "-\otimes\mathrm{CGrp}"] & \mathrm{Pr}_{L,\mathrm{add}} \arrow[l, shift left, hook'] \arrow[r, shift left, "-\otimes\mathrm{Sp}"] & \mathrm{Pr}_{L,\mathrm{st}} \arrow[l, shift left, hook']
\end{tikzcd}$$

Warning 2.5.16. The canonical presentably symmetric monoidal structures on $\mathrm{An}_*, \mathrm{CMon}, \mathrm{CGrp}, \mathrm{Sp}$ are *not* the (co)cartesian symmetric monoidal structures! For example, it is a good exercise to see that the cartesian symmetric monoidal structures on these categories cannot be presentably symmetric monoidal.

One can deduce many interesting results from Theorem 2.5.15, for example the following which will play a very important role in this course.

Corollary 2.5.17 (Symmetric monoidality of group–completions (exercise))**.** *Let* $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}_L^{\otimes})$.

(1) *The adjunction unit* $\eta\colon \mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C})$ *naturally refines to a symmetric monoidal functor between presentably symmetric monoidal categories.*

(2) *The functor* $\eta\colon \mathrm{CMon}(\mathcal{C}) \to \mathrm{CGrp}(\mathcal{C})$ *may be identified as the group–completion functor* $(-)^{\mathrm{gp}}$*, i.e. its right adjoint is given by the inclusion* $\mathrm{CGrp}(\mathcal{C}) \hookrightarrow \mathrm{CMon}(\mathcal{C})$*. In particular, the group–completion functor induces a functor on* $\mathbb{E}_\infty$*–algebras* $(-)^{\mathrm{gp}}\colon \mathrm{CAlg}(\mathrm{CMon}(\mathcal{C})^{\otimes}) \to \mathrm{CAlg}(\mathrm{CGrp}(\mathcal{C})^{\otimes})$.

The following construction, due to Bousfield in his seminal paper [Bou79], is very standard and we will have use of them for our applications chromatic redshift in the final chapter.

Construction 2.5.18 (Localization of spectra with respect to homology)**.** Let $E \in \mathrm{Sp}$. We define Sp_E to be the full subcategory of *E–local spectra*, i.e. those $Y \in \mathrm{Sp}$ such that for all $A \in \mathrm{Sp}$ with $E \otimes A \simeq 0$, we have $\mathrm{Map}_{\mathrm{Sp}}(A, Y) \simeq 0$. Such spectra A are called E–acyclic. It turns out that we have a Bousfield localisation

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, shift left, "L_E"] & \mathrm{Sp}_E \arrow[l, shift left, hook] \end{tikzcd}$$

where a morphism $X \to Y$ is an L_E–equivalence if and only if $E \otimes X \to E \otimes Y$ is an equivalence. Because of this characterisation, it is easy to see by virtue of Proposition 2.5.10 that L_E upgrades to a symmetric monoidal functor, and so the right adjoint thus upgrades to a lax symmetric monoidal functor.

Here are two very important examples of the construction above.

Example 2.5.19 (p–localisation and p–completion)*.*

(1) Using $E = \mathbb{S}_{(p)} \coloneqq \mathbb{S}[\{\text{primes not } p\}^{-1}]$, we obtain the Bousfield localisation $\mathrm{Sp} \to \mathrm{Sp}_{(p)}$ of p–localisation.

(2) Using $E = \mathbb{S}/p$, we obtain the Bousfield localisation $(-)^{\wedge}_p\colon \mathrm{Sp} \to \mathrm{Sp}^{\wedge}_p$ of p–completion.

2.6 Duality and dualisability

In the groundbreaking paper [DP84], among other things, Dold and Puppe axiomatised the notion of duality, providing a far–reaching generalisation of dual vector spaces. We summarise this notion now in the modern setting.

Definition 2.6.1 ([Lur17, Def. 4.6.1.1])**.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal ∞–category. A *duality datum* in $\mathcal{C}$ is a tuple $(X, X^\vee, c, e)$ where $X, X^\vee \in \mathcal{C}$ and c, e are morphisms

$$c\colon \mathbb{1} \to X \otimes X^\vee \qquad e\colon X^\vee \otimes X \to \mathbb{1}$$

such that the composites

$$X \xrightarrow{c\otimes\mathrm{id}} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}\otimes e} X$$

$$X^\vee \xrightarrow{\mathrm{id}\otimes c} X^\vee \otimes X \otimes X^\vee \xrightarrow{e\otimes \mathrm{id}} X^\vee$$

are homotopic to the identity morphisms. An object $X \in \mathcal{C}$ which participate in such a datum is said to be *dualisable.*

Observation 2.6.2. Since duality data are tuples satisfying homotopy conditions, they are really notions that are controlled in the homotopy category with the induced 1–symmetric monoidal structure $(\mathrm{Ho}(\mathcal{C}), \otimes, \mathbb{1})$. That is, a tuple $(X, X^\vee, c, e)$ is a duality datum in $(\mathcal{C}, \otimes, \mathbb{1})$ if and only if it is one in $(\mathrm{Ho}(\mathcal{C}), \otimes, \mathbb{1})$.

There is an alternative characterisation of duality data that is often used interchangeably when talking about dualisable objects. The proof is a straightforward unwinding of definitions which we will omit in this note.

Proposition 2.6.3 ([Lur17, Lem. 4.6.1.6])**.** *If an object $X \in \mathcal{C}$ is dualisable with dual $X^\vee$, then the duality datum provides a natural equivalence of functors*

$$\mathrm{Map}_{\mathcal{C}}(-, X \otimes -) \simeq \mathrm{Map}_{\mathcal{C}}(- \otimes X^\vee, -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{An}$$

Observation 2.6.4. In particular, when $(\mathcal{C}, \otimes, \mathbb{1})$ is presentably symmetric monoidal stable, we have an equivalence of functors $\mathrm{Hom}_{\mathcal{C}}(X, -) \simeq X^\vee \otimes -$. This in particular implies the special property that $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves small colimits, which is very much not true in general.

Finally, we record the following "descent" property of the notion of dualisability.

Proposition 2.6.5 (Descent for dualisability, [Lur17, Prop. 4.6.1.11])**.** *Let $\{\mathcal{C}_i\}_{i\in I}$ be a diagram of symmetric monoidal ∞–categories with limit $\mathcal{C}$. In this case, an object in $\mathcal{C}$ is dualisable if and only if its image in each $\mathcal{C}_a$ is dualisable.*

While we are mainly interested in $\mathbf{E}_\infty$–structures, we will often also need to talk about $\mathbf{E}_1$–structures. These are the higher algebraic analogue of associative algebras, relevant examples of which include group rings $R[G]$ (that we will see later) as well as the example in the famous result of Schwede–Shipley which says:

Theorem 2.6.6 (Schwede–Shipley, [Lur17, Thm. 7.1.2.1], [HW21, Thm. II.58])**.** *Let $\mathcal{C}$ be a stable presentable ∞–category. If it has a compact generator $X \in \mathcal{C}$, then* $\mathrm{map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \to \mathrm{Sp}$ *lifts to an equivalence*

$$\mathrm{map}_{\mathcal{C}}(X, -) \colon \mathcal{C} \xrightarrow{\simeq} \mathrm{RMod}_{\mathrm{Sp}}(\mathrm{End}(X))$$

where End(X) *is the $\mathbf{E}_1$–ring spectrum with underlying object* $\mathrm{map}_{\mathcal{C}}(X, X)$ *and multiplication given by composition.*

3 Group–completion K–theory

3.1 Definition and the group–completion theorem

Having set up all the requisite category theory, we may finally give a definition of the first variant of K–theory that we shall study in this course. As a warm–up, we give here two exercises which will give some intuition for the homotopy groups of objects in CMon(An) and CGrp(An).

Exercise 3.1.1 (Well–definedness of higher k–groups)**.** Show that every path–component of an object $M \in \mathrm{CGrp}(\mathrm{An})$ are equivalent. That is, for every $x, y \in \pi_0 M$ and writing M_x, M_y for the components of x and y in M respectively, we have an equivalence $M_x \simeq M_y$. In particular, it makes sense to speak of $\pi_i M$ without having to specify which path–component we are taking for the higher π_i's.

Exercise 3.1.2. Show that for any $M \in \mathrm{CMon}(\mathrm{An})$, $\pi_1(M, 0)$ is always abelian.

Here is the definition of the first variant of K–theory we will study in this course.

Definition 3.1.3 (Group–completion K–theory)**.** The *group–completion K–theory functor* k is defined to be the composite

$$\mathrm{k}\colon\ \mathrm{CMon}(\mathrm{Cat}) \xrightarrow{(-)^{\simeq}} \mathrm{CMon}(\mathrm{An}) \xrightarrow{(-)^{\mathrm{gp}}} \mathrm{CGrp}(\mathrm{An})$$

For $i \in \mathbb{Z}_{\geq 0}$ and $\mathcal{C} \in \mathrm{CMon}(\mathrm{Cat})$, we write $\mathrm{k}_i(\mathcal{C}) \coloneqq \pi_i \mathrm{k}(\mathcal{C})$ and refer to these as the *k–groups of* $\mathcal{C}$.

Remark 3.1.4. Since $\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}$ is limit–preserving, it is in particular product–preserving and so we may also apply the group–completion K–theory functor to symmetric monoidal 1–categories via $\mathrm{CMon}(\mathrm{Cat}_1) \hookrightarrow \mathrm{CMon}(\mathrm{Cat}) \xrightarrow{\mathrm{k}} \mathrm{CGrp}(\mathrm{An})$.

Construction 3.1.5 (Symmetric monoidal refinement of group–completion K–theory)**.** By [GGN15, Lem. 6.1], the functor $(-)^{\simeq}\colon \mathrm{CMon}(\mathrm{Cat}) \to \mathrm{CMon}(\mathrm{An})$ has a canonical lax symmetric monoidal refinement. Therefore, since $(-)^{\mathrm{gp}}$ has a symmetric monoidal refinement by Corollary 2.5.17, we obtain all in all a lax symmetric monoidal refinement of the functor k. In particular, this induces a functor

$$\mathrm{k}\colon \mathrm{CAlg}(\mathrm{CMon}(\mathrm{Cat})^{\otimes}) \longrightarrow \mathrm{CAlg}(\mathrm{CGrp}(\mathrm{An})^{\otimes})$$

Example 3.1.6 (k–theory of classical commutative rings). Let $R \in \mathrm{CRing}$ and write Proj_R for the 1–category of finitely generated projective R–modules. We define $\mathrm{k}(R) \coloneqq \mathrm{k}(\mathrm{Proj}_R)$. By Construction 3.1.5, since Proj_R is naturally an object in $\mathrm{CAlg}(\mathrm{CMon}(\mathrm{Cat}_1)^{\otimes})$ using the direct sum and tensor product operations, we see that $\mathrm{k}(R)$ is naturally an $\mathbb{E}_\infty$–algebra in $\mathrm{CGrp}(\mathrm{An})$.

Construction 3.1.7 (Localisations of commutative monoids, [Nik17, Cons. 4], [HW21, III.11a])**.** Let $M \in \mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CAlg}(\mathrm{An}^{\times})$. Then the functor $\mathrm{fgt}\colon \mathrm{LMod}_M(\mathrm{An}^{\times}) \to \mathrm{An}$ preserves colimits. Moreover, for any $N \in \mathrm{LMod}_M(\mathrm{An}^{\times})$, multiplying N by any element of $\pi_0 M$ is a morphism of left M–modules. Therefore, all in all, we may construct a "telescopic localisation" $N[M^{-1}]$ of any $N \in \mathrm{LMod}(\mathrm{An}^{\times})$ as follows: for $s \in \pi_0 M$, we define $N_s \coloneqq \mathrm{colim}(N \xrightarrow{s} N \xrightarrow{s} N \xrightarrow{s} \cdots)$. For $T = \{t_1, \ldots, t_n\} \subseteq \pi_0 M$ a finite subset, we define inductively $N_T \coloneqq (N_{t_1,\ldots,t_{n-1}})_{t_n}$. We then define

$$N[N^{-1}] \coloneqq \underset{T \subseteq \pi_0 M \text{ finite}}{\mathrm{colim}} N_T$$

Warning 3.1.8. In general, the localisation constructed above does *not* have the expected universal property. In fact, it is not even always true that for every $m \in M$, m acts invertibly on $M[M^{-1}]$! We will study this subtle phenomenon in more detail in §3.2.

Construction 3.1.9 (Localisations of $\mathbb{E}_\infty$ rings)**.** Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $S \subseteq \pi_0 R$ a multiplicative subset. Then there is a Bousfield localisation

$$\begin{tikzcd} \mathrm{Mod}_R(\mathrm{Sp}) \arrow[r, shift left, "S^{-1}"] & \mathrm{Mod}_R^{S^{-1}}(\mathrm{Sp}) \arrow[l, shift left, hook'] \end{tikzcd}$$

where $\mathrm{Mod}_R^{S^{-1}}(\mathrm{Sp}) \subseteq \mathrm{Mod}_R(\mathrm{Sp})$ is the full subcategory of R–modules such that S acts invertibly. Concretely, the functor S^{-1} is given by a "telescopic localisation" in the same way as in Construction 3.1.7. Observe that unlike the case of Construction 3.1.7, the localisation here *does* have the expected universal property of yielding the Bousfield localisation above. One can check easily that:

- the functor S^{-1} above is equivalently described as $S^{-1}R \otimes_R -$,
- $R \to S^{-1}R$ is an idempotent object.

Hence, by Proposition 2.5.13, we see that $S^{-1}R$ canonically refines to an object in $\mathrm{CAlg}(\mathrm{Mod}_R)$ and the Bousfield localisation above refines to a symmetric monoidal functor (and so its right adjoint is lax symmetric monoidal). In particular, it induces the following Bousfield localisation

$$\begin{tikzcd} \mathrm{CAlg}_{R/} \simeq \mathrm{CAlg}(\mathrm{Mod}_R(\mathrm{Sp})^{\otimes_R}) \arrow[r, shift left, "S^{-1}"] & \mathrm{CAlg}(\mathrm{Mod}_R^{S^{-1}}(\mathrm{Sp})) \simeq \mathrm{CAlg}_{R/}^{S^{-1}} \arrow[l, shift left, hook'] \end{tikzcd}$$

where the equivalences are by a standard theorem of Lurie. See [HW21, Cor. III.4a] for more details.

Notation 3.1.10. Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ and $S \subseteq \pi_0 R$ a multiplicative subset. We write $\mathrm{Map}^{S^{-1}}_{\mathrm{CAlg}}(R,-) \subseteq \mathrm{Map}_{\mathrm{CAlg}}(R,-)$ for the subcomponents of morphisms that sent elements in S to units. Note that the localisation map $\eta\colon R \to S^{-1}R$ induces a map $\eta^*\colon \mathrm{Map}_{\mathrm{CAlg}}(S^{-1}R,-) \to \mathrm{Map}^{S^{-1}}_{\mathrm{CAlg}}(R,-)$.

We learnt of the following proof from [HW21, Cor. III.4a].

Lemma 3.1.11. *Let $R, A \in \mathrm{CAlg}(\mathrm{Sp})$ and $S \subseteq \pi_0 R$ a multiplicative subset. Then the map $\eta\colon R \to S^{-1}R$ induces an equivalence of mapping anima*

$$\eta^*\colon \mathrm{Map}_{\mathrm{CAlg}}(S^{-1}R, A) \longrightarrow \mathrm{Map}^{S^{-1}}_{\mathrm{CAlg}}(R, A)$$

Proof. We just have to show that for every $f \in \pi_0 \mathrm{Map}^{S^{-1}}_{\mathrm{CAlg}}(R, A)$, the pullback

$$\{f\} \times_{\mathrm{Map}^{S^{-1}}_{\mathrm{CAlg}}(R,A)} \mathrm{Map}_{\mathrm{CAlg}}(S^{-1}R, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R/}}(S^{-1}R, A)$$

is contractible. Here, in the second equivalence, we have used the choice of the map $f\colon R \to A$ to view A as an R–algebra. Since f sends S to units, A is an R–algebra on which S acts invertibly. Thus we see by the universal property of $S^{-1}R$ that $\mathrm{Map}_{\mathrm{CAlg}_{R/}}(S^{-1}R, A) \simeq \mathrm{Map}_{\mathrm{CAlg}_{R/}}(R, A)$. But $R \in \mathrm{CAlg}_{R/}$ is the initial object, and so $\mathrm{Map}_{\mathrm{CAlg}_{R/}}(R, A) \simeq *$, as was to be shown. □

Observation 3.1.12. By Theorem 2.5.15, we see that the left adjoint $\mathbb{S}[-] = \Sigma^\infty_+\colon \mathrm{An} \to \mathrm{Sp}$ to the functor $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ canonically refines to a symmetric monoidal functor. In particular, upon applying CAlg, we obtain the adjunction

$$\mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CAlg}(\mathrm{An}^\times) \underset{\Omega^\infty}{\overset{\mathbb{S}[-]}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sp}^\otimes) \tag{3.1}$$

and so for $M \in \mathrm{CMon}(\mathrm{An})$, the spectrum $\mathbb{S}[M] \in \mathrm{Sp}$ attains a natural commutative algebra structure.

Construction 3.1.13. Since $M \to M^{\mathrm{gp}}$ is a map of commutative algebras in $\mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CAlg}(\mathrm{An}^\times)$, we see that the elements in $\pi_0 M$ act invertibly on M^{gp}. By the functorial telescopic localisation from Construction 3.1.7, we obtain a canonical map $M[M^{-1}] \to M^{\mathrm{gp}}[M^{-1}]$. But then since all the morphisms in the telescopic localisation constructing $M^{\mathrm{gp}}[M^{-1}]$ are equivalences, we see that $M^{\mathrm{gp}}[M^{-1}] \simeq M^{\mathrm{gp}}$. Thus, all in all, we obtain a canonical map

$$\mathrm{can}\colon M[M^{-1}] \longrightarrow M^{\mathrm{gp}} \tag{3.2}$$

Moreover, if we write $\pi_M \subseteq \pi_0 \mathbb{S}[M]$ for the image of the Hurewicz map $\pi_0 M \to \pi_0 \Omega^\infty \mathbb{S}[M]$, then since $\mathbb{S}[-]\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CAlg}(\mathrm{Sp}^\otimes)$ is a left adjoint, it preserves colimits and we see that

$$\mathbb{S}\big[M[M^{-1}]\big] \simeq \pi_M^{-1}\mathbb{S}[M]$$

Our presentation of the classical group–completion theorem is based on Nikolaus' simple proof in [Nik17]. To avoid talking about the left Ore conditions, we will only consider the $\mathbb{E}_\infty$ version in this course.

Theorem 3.1.14 (Group completion theorem)**.** *Let* $M \in \mathrm{CMon}(\mathrm{An})$*. Let* $\pi_M \subseteq \pi_0 \mathbb{S}[M]$ *be the image of the Hurewicz map* $\pi_0 M \to \pi_0 \Omega^\infty \mathbb{S}[M]$*. Then the canonical map* $M \to M^{\mathrm{gp}}$ *induces an equivalence in* $\mathrm{CAlg}(\mathrm{Sp})$

$$\pi_M^{-1}\mathbb{S}[M] \longrightarrow \mathbb{S}[M^{\mathrm{gp}}]$$

Equivalently, the canonical map $M[M^{-1}] \to M^{\mathrm{gp}}$ *from (3.2) induces an equivalence* $\mathbb{S}\big[M[M^{-1}]\big] \to \mathbb{S}[M^{\mathrm{gp}}]$*.*

Proof. The second equivalent statement is an immediate consequence of the first statement. To see the first statement, we claim that $M^{\mathrm{gp}} \in \mathrm{CMon}(\mathrm{An})$ satisfies the following universal property: for every $X \in \mathrm{CMon}(\mathrm{An})$, the map

$$\mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}(M^{\mathrm{gp}}, X) \to \mathrm{Map}^{(\pi_0 M)^{-1}}_{\mathrm{CMon}(\mathrm{An})}(M, X)$$

induced by $\eta\colon M \to M^{\mathrm{gp}}$ is an equivalence, where $\mathrm{Map}^{(\pi_0 M)^{-1}}_{\mathrm{CMon}(\mathrm{An})} \subseteq \mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}$ means the subcomponents of maps where $\pi_0 M$ is sent to elements that admit additive inverses in $\pi_0 X$. The map lands in these subcomponents since M^{gp} is group–complete. To prove the claim, define $X^\times$ as the pullback in $\mathrm{CMon}(\mathrm{An})$

$$\begin{tikzcd}
X^\times \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
(\pi_0 X)^\times \arrow[r, hook] & \pi_0 X
\end{tikzcd}$$

so that $X^\times \in \mathrm{CGrp}(\mathrm{An})$. Now consider the commuting diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}(M^{\mathrm{gp}}, X) \arrow[r, "\eta^*"] & \mathrm{Map}^{(\pi_0 M)^{-1}}_{\mathrm{CMon}(\mathrm{An})}(M, X) \\
\mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}(M^{\mathrm{gp}}, X^\times) \arrow[u, "i_*"] \arrow[r, "\eta^*"'] & \mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}(M, X^\times) \arrow[u, "i_*"']
\end{tikzcd}$$

The left vertical i_* is an equivalence since M^{gp} is group–complete and $(-)^\times$ is the right adjoint to the inclusion $\mathrm{CGrp}(\mathrm{An}) \subseteq \mathrm{CMon}(\mathrm{An})$; the bottom η^* is an equivalence since $X^\times$ is group–complete and M^{gp} is the group–completion of M; the right vertical i_* is an equivalence because maps in $\mathrm{Map}^{(\pi_0 M)^{-1}}$ are precisely those that land in $X^\times$ by definition. Therefore, all in all, the top horizontal η^* is also an equivalence, as claimed.

Now by the adjunction (3.1), for any $A \in \mathrm{CAlg}(\mathrm{Sp})$, we have

$$\begin{aligned}
\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}[M^{\mathrm{gp}}], A) &\simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{An})}(M^{\mathrm{gp}}, \Omega^\infty A) \\
&\simeq \mathrm{Map}^{(\pi_0 M)^{-1}}_{\mathrm{CMon}(\mathrm{An})}(M, \Omega^\infty A) \\
&\simeq \mathrm{Map}^{(\pi_M)^{-1}}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathbb{S}[M], A) \\
&\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\pi_M^{-1}\mathbb{S}[M], A)
\end{aligned} \tag{3.3}$$

where the second equivalence is by the claim above and the last equivalence is by Lemma 3.1.11. This finishes the proof of the theorem. ☐

Notation 3.1.15. For a spectrum E and an anima X, we write $H_*(X;E) \coloneqq \pi_*(\mathbb{S}[X]\otimes E)$. In the case when E is the Eilenberg–Mac Lane spectrum for some classing ring R, this definition agrees with the singular homology of X with R coefficient.

Corollary 3.1.16 (Homological group–completion)**.** *Let* $M \in \mathrm{CMon}(\mathrm{An})$*,* $R \in \mathrm{CAlg}(\mathrm{Sp})$*, and* $E \in \mathrm{Mod}_R(\mathrm{Sp})$*. Then there is an isomorphism of* π_*R*–modules*

$$\pi_M^{-1}H_*(M;E) \xrightarrow{\cong} H_*(M^{\mathrm{gp}};E)$$

Proof. Just apply the colimit–preserving functor $E\otimes -$ to the equivalence $\pi_M^{-1}\mathbb{S}[M] \longrightarrow \mathbb{S}[M^{\mathrm{gp}}]$ and apply π_*. ☐

Notation 3.1.17. The inclusion $\mathrm{CGrp}(\mathrm{Set}) \subseteq \mathrm{CMon}(\mathrm{Set})$ also has a left adjoint, which we denote by $(-)^{\mathrm{gp}_0}$.

Observation 3.1.18 (k_0 groups)*.* Since the left square in

$$\begin{tikzcd}
\mathrm{CMon} & \mathrm{CGrp} \arrow[l, hook'] & & \mathrm{CMon} \arrow[r, "(-)^{\mathrm{gp}}"] \arrow[d, "\pi_0"] & \mathrm{CGrp} \arrow[d, "\pi_0"] \\
\mathrm{CMon(Set)} \arrow[u, hook] & \mathrm{CGrp(Set)} \arrow[l, hook'] \arrow[u, hook] & & \mathrm{CMon(Set)} \arrow[r, "(-)^{\mathrm{gp}_0}"] & \mathrm{CGrp(Set)}
\end{tikzcd}$$

clearly commutes, we also know by passing to left adjoints that the right square above commutes. That is, for $M \in \mathrm{CMon}$, the canonical comparison map

$$(\pi_0 M)^{\mathrm{gp}_0} \xrightarrow{\simeq} \pi_0(M^{\mathrm{gp}})$$

is an equivalence. Thus, we obtain an isomorphism

$$\mathrm{k}_0(\mathcal{C}) \cong \big(\pi_0(\mathcal{C}^{\simeq})\big)^{\mathrm{gp}_0}$$

computing the zeroth k–group.

A priori, the preceding observation does not give us that the two different group–completions agree on commutative monoids in sets. However, as we shall show in the next result, this is indeed true by virtue of the previous proposition.

Corollary 3.1.19 (Group–completions of abelian monoids)**.** *Let* $M \in \mathrm{CMon}(\mathrm{Set})$*. Then* $M^{\mathrm{gp}} \simeq M^{\mathrm{gp}_0}$*.*

Proof. First note that since homotopy groups commute with filtered colimits, we see that $M[M^{-1}] \in \mathrm{Set} \subseteq \mathrm{An}$. In particular, $M[M^{-1}]$ is a simple anima because it is simply–connected, and M^{gp} is a simple anima because it has the structure of an $\mathbb{E}_\infty$–monoid. Now, by the group–completion Theorem 3.1.14, the canonical map $M[M^{-1}] \to M^{\mathrm{gp}}$ induces an isomorphism of $\mathbb{Z}$–homology. And so by the homology Whitehead theorem for simple anima, this map is an equivalence and in particular, $M^{\mathrm{gp}} \simeq \pi_0(M^{\mathrm{gp}})$. Thus $M^{\mathrm{gp}} \simeq \pi_0(M^{\mathrm{gp}}) \simeq (\pi_0 M)^{\mathrm{gp}_0}$ where the second equivalence is by Observation 3.1.18. ☐

Example 3.1.20. By the corollary above, we see for example that $\mathbb{N}^{\mathrm{gp}} \simeq \mathbb{N}^{\mathrm{gp}_0} \simeq \mathbb{Z}$.

Remark 3.1.21. It was crucial in Corollary 3.1.19 that the discrete monoid was abelian. Otherwise, there is a very strong "no–go" result due to McDuff which says that every connected anima is equivalent to BM for some discrete monoid M.

Next, recalling the definition of $\mathrm{k}(R)$ for a classical commutative ring R from Example 3.1.6, we show how to use the group–completion Theorem 3.1.14 to compute its homology and $\mathrm{k}_1(R)$.

Corollary 3.1.22 (Homology of $\mathrm{k}(R)$)**.** *Let $R \in \mathrm{CRing}$. Then there is a canonical isomorphism*

$$H_*(\mathrm{GL}_\infty(R);\mathbb{Z}) \cong \operatorname*{colim}_{n\in\mathbb{N}} H_*(\mathrm{GL}_n(R);\mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{k}(R)_0;\mathbb{Z})$$

Proof. First, observe that we have an equivalence of 1–groupoids $\mathrm{Proj}(R)^{\simeq} \simeq \coprod_{P\in\mathrm{Proj}(R)} B\mathrm{Aut}_R(P)$. As a matter of notation, note that $\mathrm{Aut}_R(R^{\oplus n}) = \mathrm{GL}_n(R)$. Moreover, since for every finitely generated projective R–module P, there exists a finitely generated projective R–module Q such that $P \oplus Q \cong R^{\oplus n}$ for some n, we see by a cofinality argument that $\left(\coprod_{P\in\mathrm{Proj}(R)} B\mathrm{Aut}_R(P)\right)\left[\left(\coprod_{P\in\mathrm{Proj}(R)} B\mathrm{Aut}_R(P)\right)^{-1}\right]$ is equivalent to the colimit

$$C \coloneqq \operatorname*{colim}_{n\in\mathbb{N}} \Big(\coprod_{P\in\mathrm{Proj}(R)} B\mathrm{Aut}_R(P) \xrightarrow{+R} \coprod_{P\in\mathrm{Proj}(R)} B\mathrm{Aut}_R(P) \xrightarrow{+R} \cdots \Big)$$

By Corollary 3.1.16, we learn that the map $C \to \mathrm{k}(R)$ induces an isomorphism

$$H_*(C;\mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{k}(R);\mathbb{Z}) \tag{3.4}$$

Now we need to isolate the appropriate components: note first that the component $* \simeq B\mathrm{Aut}_R(0) \subseteq \mathrm{Proj}_R^{\simeq}$ is mapped to $\mathrm{k}(R)_0$. As explained in the construction of the canonical map $\mathrm{Proj}_R^{\simeq} \to \mathrm{k}(R)$ in (3.2), the localising maps on $\mathrm{k}(R)$ are all equivalences, and so we obtain in total that the map $C \to \mathrm{k}(R)$ restricts to a map

$$\operatorname*{colim}_{n\in\mathbb{N}} \big(B\mathrm{GL}_0(R) \xrightarrow{+R} B\mathrm{GL}_1(R) \xrightarrow{+R} B\mathrm{GL}_2(R) \xrightarrow{+R} \cdots\big) \longrightarrow \mathrm{k}(R)_0$$

on the appropriate components. Thus, the homology isomorphism (3.4) restricts to the homology isomorphism as in the statement of the corollary. ☐

Corollary 3.1.23 (k_1–groups)**.** *Let $R \in \mathrm{CRing}$. We have an isomorphism*

$$\mathrm{k}_1(R) \cong \mathrm{GL}_\infty(R)^{\mathrm{ab}}.$$

Moreover, if R were a field, then $\mathrm{GL}_\infty(R) \cong R^\times$.

Proof. That $\mathrm{GL}_\infty(R) \cong R^\times$ in the case of fields is a linear algebra fact that may be found for example in [Wei13, Example III.1.3.5]. For the isomorphism $\mathrm{k}_1(R) \cong \mathrm{GL}_\infty(R)^{\mathrm{ab}}$, recall the standard algebraic topology fact that for a connected anima

X, $H_1(X;\mathbb{Z}) \cong \pi_1(X)^{\mathrm{ab}}$. Moreover, by Exercise 3.1.2, $\mathrm{k}_1(R)$ is abelian. Thus, from Corollary 3.1.22, we see that

$$\mathrm{k}_1(R) \cong \mathrm{k}_1(R)^{\mathrm{ab}} \cong H_1(\mathrm{k}(R)_0;\mathbb{Z}) \cong H_1(\mathrm{GL}_\infty(R);\mathbb{Z}) \cong \mathrm{GL}_\infty(R)^{\mathrm{ab}}$$

as required. □

3.2 Cyclic invariance and the plus construction

From the group–completion Theorem 3.1.14, we saw that the map $M[M^{-1}] \to M^{\mathrm{gp}}$ is at least a homology isomorphism. As we shall see in an explicit Example 3.2.22, this is not always an equivalence. The goal of this subsection is to locate precisely the failure of this map to be an equivalence by way of Quillen's famous plus–construction. The first step in this direction is to come to terms with a subtle idiosyncrasy of higher algebra called the "cyclic invariance condition" which is not seen at the level of classical algebra.

Observation 3.2.1 (Data of direct systems). Note that $(\mathbb{N}, \leq) \in \mathrm{Cat}$ is also computed as the colimit

$$\Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \cdots$$

Hence, for any $\mathcal{C} \in \mathrm{Cat}$, we get

$$\mathrm{Fun}(\mathbb{N}, \mathcal{C}) \simeq \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \mathcal{C}^{\Delta^1} \times_{\mathcal{C}} \cdots$$

In particular, we see that for $M_\bullet, N_\bullet \in \mathrm{Fun}(\mathbb{N}, \mathcal{C})$, we have

$$\begin{aligned}&\mathrm{Map}_{\mathcal{C}^{\mathbb{N}}}(M_\bullet, N_\bullet)\\ &\simeq \mathrm{Map}_{\mathcal{C}^{\Delta^1}}(M_1 \to M_2, N_1 \to N_2) \times_{\mathrm{Map}_{\mathcal{C}}(M_2,N_2)} \mathrm{Map}_{\mathcal{C}^{\Delta^1}}(M_2 \to M_3, N_2 \to N_3) \times \cdots\end{aligned}$$

That is, to specify a map $M_\bullet \to N_\bullet$ in $\mathcal{C}^{\mathbb{N}}$, it suffices to provide the data of $\{f_i\}_{i\in\mathbb{N}}$ and $\{\sigma_{i,i+1}\}_{i\in\mathbb{N}}$ in the diagram

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"'] \arrow[dr, phantom, "\equiv_{\sigma_{1,2}}"] & M_2 \arrow[r] \arrow[d, "f_2"'] \arrow[dr, phantom, "\equiv_{\sigma_{2,3}}"] & M_3 \arrow[r] \arrow[d, "f_3"'] \arrow[dr, phantom, "\equiv_{\sigma_{3,4}}"] & \cdots \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 3.2.2. Let $M, A \in \mathrm{CMon}$, $n \in \mathbb{N}$, $m \in M$, and $M \to A$ a map in CMon. We may view m as an element in A by prolonging $* \xrightarrow{m} M \to A$. Show that the map $\overline{m}\colon A[m^{-1}] \to A[m^{-1}]$ induced by the diagram

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & A \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & A \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & \cdots \\
A \arrow[r, "m"] & A \arrow[r, "m"] & A \arrow[r, "m"] & \cdots
\end{tikzcd}$$

has an inverse induced by the diagram

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] & A \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "\equiv"] & A \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "\equiv"] & \cdots \\
 & A \arrow[r, "m"] & A \arrow[r, "m"] & \cdots
\end{tikzcd}$$

Hint: *note that the identity map* $A[m^{-1}] \to A[m^{-1}]$ *is induced by the commuting diagram*

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "\mathrm{can}_1"'] & A \arrow[r, "m"] \arrow[dl, "\mathrm{can}_2"] & A \arrow[r, "m"] \arrow[dll, bend left, "\mathrm{can}_3"] & \cdots \\
A[m^{-1}] & & &
\end{tikzcd}$$

Warning 3.2.3 (Failure of localisations to be local)*.* The map $\overline{m}\colon M[m^{-1}] \to M[m^{-1}]$ as in the exercise above is in general *not* equivalent to the map $[m]\colon M[m^{-1}] \to M[m^{-1}]$ induced by the diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & M \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & M \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & \cdots \\
M \arrow[r, "m"] & M \arrow[r, "m"] & M \arrow[r, "m"] & \cdots
\end{tikzcd}$$

To understand what is going on, it would be beneficial to consider to consider the following extended exercise.

Exercise 3.2.4 (Symmetric groups fail to localise properly)**.** Let us consider $M = \mathrm{Fin}^{\simeq} = \coprod_{n\geq 0} B\Sigma_n$. As a warm up,consider the situation in Exercise 3.2.2: picking a component $B\Sigma_k$ in the source, say, we end up having to consider the diagram

$$\begin{tikzcd}
B\Sigma_k \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & B\Sigma_{k+1} \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & B\Sigma_{k+1+1} \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_{\mathrm{id}}"] & \cdots \\
B\Sigma_{k+1} \arrow[r, "+1"] & B\Sigma_{k+1+1} \arrow[r, "+1"] & B\Sigma_{k+1+1+1} \arrow[r, "+1"] & \cdots
\end{tikzcd}$$

(i) Show that upon applying π_1, the induced homomorphism on the colimits $\Sigma_\infty \to \Sigma_\infty$ is given by $+1$, i.e. it sends a permutation $f(1,\ldots,k) \in \Sigma_k$ to $f(1,\ldots,k)\cdot \mathrm{id}(k+1) \in \Sigma_{k+1}$,

(ii) Show that this is an isomorphism of groups.

Now we put ourselves in the setting of Warning 3.2.3. Picking a component $B\Sigma_k$ in the source, say, we end up having to consider the diagram

$$\begin{tikzcd}
B\Sigma_k \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_\tau"] & B\Sigma_{k+1} \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_\tau"] & B\Sigma_{k+1+1} \arrow[r, "+1"] \arrow[d, "+1"'] \arrow[dr, phantom, "\equiv_\tau"] & \cdots \\
B\Sigma_{k+1} \arrow[r, "+1"] & B\Sigma_{k+1+1} \arrow[r, "+1"] & B\Sigma_{k+1+1+1} \arrow[r, "+1"] & \cdots
\end{tikzcd}$$

where τ swaps the last two copies of $1+1$. The key point is the following: applying π_1 to this diagram, we obtain a commuting diagram of groups "with a wrinkle"

$$\begin{tikzcd}
\Sigma_k \arrow[rr, "+1"] \arrow[d, "+1"'] & & \Sigma_{k+1} \arrow[rr, "+1"] \arrow[d, "+1"'] & & \Sigma_{k+1+1} \arrow[r, "+1"] \arrow[d, "+1"'] & \cdots \\
\Sigma_{k+1} \arrow[r, "+1"] & \Sigma_{k+1+1} \arrow[r, "\tau^{-1}(-)\tau", "\cong"'] & \Sigma_{k+1+1} \arrow[r, "+1"] & \Sigma_{k+1+1+1} \arrow[r, "\tau^{-1}(-)\tau", "\cong"'] & \Sigma_{k+1+1+1} \arrow[r, "+1"] & \cdots
\end{tikzcd}$$

In concrete terms, the bottom maps send for example a permutation $f(1, \ldots, k+1) \in \Sigma_{k+1}$ to $f(1, \ldots, k, k+2) \cdot \mathrm{id}(k+1) \in \Sigma_{k+1+1}$.

(a) Show using this description that the induced group homomorphism $\Sigma_\infty \to \Sigma_\infty$ is not surjective. **Hint:** *for example, letting $k = 1$, show that the element $\sigma(1, 2) \in \Sigma_2 \to \Sigma_\infty$ in the bottom is not hit by any element from the top Σ_∞.*

(b) As a sanity check, convince yourself that the bottom colimit is isomorphic to Σ_∞, for example by replacing the vertical maps with $\Sigma_{k+1} \xrightarrow{+1} \Sigma_{k+1+1} \xrightarrow[\cong]{\tau^{-1}(-)\tau} \Sigma_{k+1+1}$ and so on.

Next, we explain how the map $[m]\colon M[m^{-1}] \to M[m^{-1}]$ may alternatively be viewed as being induced by the telescopic diagram $M[m^{-1}] \xrightarrow{m} M[m^{-1}] \xrightarrow{m} \cdots$.

Observation 3.2.5. For $M \in \mathrm{CMon}$ and $m \in M$, recall from Construction 3.1.7 that $M[m^{-1}]$ is again a left M–module and so it makes to consider $(M[m^{-1}])[m^{-1}]$. Note that this is equivalent to $M[m^{-1}]$ since the diagram defining $(M[m^{-1}])[m^{-1}]$ is indexed over $\mathbb{N} \times \mathbb{N}$ whereas that for $M[m^{-1}]$ is $\mathbb{N}$, and it is an easy check that the diagonal functor $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is colimit cofinal.

Now consider the cube (where we extend infinitely horizontally to the right):

$$\begin{tikzcd}
 & M \arrow[rr, "m"] \arrow[dd, "can"] \arrow[dr, phantom, "\equiv_\tau"] & & M \arrow[dd, "can"] & & \cdots \arrow[rr] & & M[m^{-1}] \arrow[dd, "can", "\simeq"'] \\
M \arrow[ur, "m"] \arrow[rr, "m"] \arrow[dd, "can"'] & & M \arrow[ur, "m"] \arrow[dd, "can"'] & & \cdots \arrow[rr] & & M[m^{-1}] \arrow[ur, "{[m]}"] \arrow[dd, "can", "\simeq"'] & \\
 & M[m^{-1}] \arrow[rr, "m"] \arrow[dr, phantom, "\equiv_\tau"] & & M[m^{-1}] & & \cdots \arrow[rr] & & (M[m^{-1}])[m^{-1}] \\
M[m^{-1}] \arrow[ur, "m"] \arrow[rr, "m"] & & M[m^{-1}] \arrow[ur, "m"] & & \cdots \arrow[rr] & & (M[m^{-1}])[m^{-1}] \arrow[ur, "{[m]}"] &
\end{tikzcd}$$

The cubical diagram commutes by virtue of the fact that $M[m^{-1}]$ is canonically a left M–module and that the canonical map $M \to M[m^{-1}]$ is a map of left M–modules. Taking horizontal colimits yields the square on the right where the vertical morphisms are equivalences by the previous paragraph. Thus, all in all, we see that the map $[m]\colon M[m^{-1}] \to M[m^{-1}]$ induced by the top diagram is an equivalence if and only if the map equivalent map induced by the bottom diagram is an equivalence.

This counterintuitive and subtle fact that the natural map $[m]\colon M[m^{-1}] \to M[m^{-1}]$ might not be an equivalence turns out to have a fix in the form of a *cyclic invariance condition*, which we prove in Lemma 3.2.7. The proof is an instructive one exhibiting the subtleties of higher algebra. To this end, we first need to construct a Σ_n–action on the set of loops on an element $m^n = m \cdot m \cdots m$:

Construction 3.2.6. Let $M, A \in \mathrm{CMon}$, $n \in \mathbb{N}$, $m \in M$, and $M \to A$ a map in CMon. We may view m as an element in A by prolonging $* \xrightarrow{m} M \to A$. We now construct a map

$$\Sigma_n \longrightarrow \pi_1(A, m^n)$$

witnessing the action of Σ_n swapping the copies of m in $m^n = m{\cdot}m{\cdot}m \cdots m$. Firstly, the choice of the tuple $(m, \ldots, m)$ in $A^{\times n}$ may be written as a map $(m, \ldots, m)\colon * \to A^{\times n}$ and applying $(-)_{h\Sigma_n}$ to this map yields a map $B\Sigma_n = *_{h\Sigma_n} \to (A^{\times n})_{h\Sigma_n}$. On the other hand, we have the multiplication map $\mu\colon A^{\times n} \to A$. Since A was an $\mathbb{E}_\infty$–monoid, this multiplication refines to a map of Σ_n–equivariant objects with the swap action on the source and trivial action on the domain. Thus, by the adjunction $(-)_{h\Sigma_n}\colon \mathrm{An}^{B\Sigma_n} \rightleftharpoons \mathrm{An} : \mathrm{trivial}_{\Sigma_n}$, we have the datum of a map $\mu_{h\Sigma_n}\colon (A^{\times n})_{h\Sigma_n} \to A$ and a commuting triangle

$$\begin{tikzcd}
A^{\times n} \arrow[r, "\mu"] \arrow[d, "\mathrm{can}"'] & A \\
(A^{\times n})_{h\Sigma_n} \arrow[ur] &
\end{tikzcd}$$

The desired map $\Sigma_n \to \pi_1(A, m^n)$ is now obtained by applying π_1 to the composition

$$B\Sigma_n \longrightarrow (A^{\times n})_{h\Sigma_n} \xrightarrow{\mu_{h\Sigma_n}} A$$

Lemma 3.2.7 (Cyclic invariance condition, [Nik17, Proof of (7) ⇒ (1) in Prop. 6])**.** *Let $M \in \mathrm{CMon}(\mathrm{An})$. Suppose that for every $m \in M$, there is an $n \geq 2$ such that the map*

$$\Sigma_n \longrightarrow \pi_1(M, m^n) \longrightarrow \pi_1(M[M^{-1}], m^n)$$

has the permutation $(123 \cdots n)$ in its kernel. Then $M[M^{-1}]$ is $\pi_0 M$–local, i.e. $\pi_0 M$ acts invertibly on $M[M^{-1}]$.

Proof. Clearly, we may assume without loss of generality that the set of generators $\{m_i\}_{i \in I}$ of $\pi_0 M$ consists of a single element $m \in \pi_0 M$.

To begin with, by Observation 3.2.5, to check that the multiplication map $[m]\colon M[m^{-1}] \to M[m^{-1}]$ is an equivalence, it is equivalent to check that the map induced by the diagram

$$\begin{tikzcd}
M[m^{-1}] \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & M[m^{-1}] \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & M[m^{-1}] \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\equiv_\tau"] & \cdots \\
M[m^{-1}] \arrow[r, "m"] & M[m^{-1}] \arrow[r, "m"] & M[m^{-1}] \arrow[r, "m"] & \cdots
\end{tikzcd}$$

is an equivalence. In order to show that this is an equivalence, we will construct an inverse by showing that the obvious candidate $[\mathrm{id}]\colon M[m^{-1}] \to M[m^{-1}]$ induced by

$$\begin{tikzcd}
M[m^{-1}] \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] & M[m^{-1}] \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "\equiv"] & M[m^{-1}] \arrow[r, "m"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "\equiv"] & \cdots \\
& M[m^{-1}] \arrow[r, "m"] & M[m^{-1}] \arrow[r, "m"] & \cdots
\end{tikzcd}$$

works. To this end, first note that either compositions $[m] \circ [\mathrm{id}]$ or $[\mathrm{id}] \circ [m]$ are given by the map induced from the diagram

$$\begin{tikzcd}
M[m^{-1}] \arrow[r, "m"] \arrow[dr, "m"'] & M[m^{-1}] \arrow[r, "m"] \arrow[dr, "m"'] \arrow[d, phantom, "\equiv_\tau"] & M[m^{-1}] \arrow[r, "m"] \arrow[dr, "m"'] \arrow[d, phantom, "\equiv_\tau"] & \cdots \\
& M[m^{-1}] \arrow[r, "m"] & M[m^{-1}] \arrow[r, "m"] & \cdots
\end{tikzcd} \tag{3.5}$$

Thus, by virtue of Exercise 3.2.2, if we could show that the 2–cells in (3.5) are equivalent to the identity maps, we would be done. Note that the cell τ being the identity path means that the permutation (12) lies in the kernel of the map $\Sigma_2 \to \pi_1(M[m^{-1}], m^2)$. In fact though, less is sufficient: we do not need to consider the 2–cells in (3.5) itself but can consider just the $(n-1)$–fold horizontal composition of the 2–cells with itself by cofinality. But the $(n-1)$–fold composition in the diagram is given by left multiplication by the symmetry path $\tau_{1,2} \circ \tau_{2,3} \circ \cdots \circ \tau_{n-1,n}$. This corresponds to the permutation $(123 \cdots n)$. Thus, it suffices that $(123 \cdots n)$ is in the kernel of the map $\Sigma_n \to \pi_1(M, m^n) \to \pi_1(M[m^{-1}], m^n)$, which is the hypothesis. This completes the proof. □

Next, we would like to see how this cyclic invariance condition can explain the failure of $M[M^{-1}] \to M^{\mathrm{gp}}$ to be an equivalence by giving various equivalent criteria for when this map is an equivalence in Proposition 3.2.10. To state the result, we will need to recall the following notion from group theory.

Recollections 3.2.8. A group G is said to be *hypoabelian* if it has no nontrivial perfect subgroups. The following are equivalent conditions for a group P to be perfect:

1. The abelianisation P^{ab} of P is the trivial group,
2. Any group homomorphism $P \to A$ where A is an abelian group is the trivial hommormophism,
3. For any element $g \in P$, there is a sequence of elements $\{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$ for some n such that g may be written as a product of commutators $[a_1, b_1][a_2, b_2] \cdots [a_n, b_n]$.

Note that abelian groups are hypoabelian.

Notation 3.2.9. Write $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ for the full subcategory of spaces with hypoabelian fundamental groups.

Proposition 3.2.10 (Localisation model of group–completion, [Nik17, Prop. 6])**.** *For the left M–space $M[M^{-1}]$, the following are equivalent:*

(1) $M[M^{-1}]$ is $\pi_0 M$–local, that is, $\pi_0 M$ acts invertibly on $M[M^{-1}]$,

(2) The map $M \to M[M^{-1}]$ exhibits the target as the universal $\pi_0 M$–local space,

(3) *The canonical map* $M[M^{-1}] \to M^{\mathrm{gp}}$ *is an equivalence,*

(4) *The fundamental groups of all components of* $M[M^{-1}]$ *are abelian,*

(5) *The fundamental groups of all components of* $M[M^{-1}]$ *are hypoabelian,*

(6) *For every* m_i*, the map*

$$\Sigma_3 \longrightarrow \pi_1(M, m_i^3) \longrightarrow \pi_1(M[M^{-1}], m_i^3)$$

has (123) *in its kernel,*

(7) *For every* m_i*, there is an* $n \geq 2$ *such that the map*

$$\Sigma_n \longrightarrow \pi_1(M, m_i^n) \longrightarrow \pi_1(M[M^{-1}], m_i^n)$$

has the permutation $(123 \cdots n)$ *in its kernel.*

Proof. That (7) implies (1) is precisely Lemma 3.2.7. The rest are left as an exercise for the reader. For some of the directions, you may use the fact that the abelianisation of Σ_3 is C_2 and the hypoabelianisation of Σ_n is given by C_2 when $n \geq 5$. ☐

Corollary 3.2.11 ([Nik17, Cor. 7])**.** *Suppose* $M \in \mathrm{CMon(An)}$ *is a hypoabelian anima. Then the map* $M[M^{-1}] \to M^{\mathrm{gp}}$ *is an equivalence. In particular, the fundamental groups of all components of* $M[M^{-1}]$ *are abelian.*

Proof. We use Proposition 3.2.10 (7). It is a fact that for $n \geq 5$ odd, the hypoabelianisation of Σ_n is C_2 with $(123 \cdots n)$ in the kernel, and so our hypothesis on M guarantees point (7) in the proposition which demands that $\Sigma_n \to \pi_1(M, m^n) \to \pi_1(M[M^{-1}], m^n)$ for some n. ☐

Remark 3.2.12. We could also use Proposition 3.2.10 to prove Corollary 3.1.19. To wit, since $\pi_1 M = 0$ in this case, this is an immediate consequence of (3) = (6) in Proposition 3.2.10.

From Proposition 3.2.10, we see that the obstruction for $M[M^{-1}]$ to compute the group–completion of M is the possibility that $M[M^{-1}]$ have fundamental groups that are not hypoabelian. But there is a universal way of fixing this introduced by Quillen, and we close this subsection with it. This will require a quick detour into the world of group theory.

Exercise 3.2.13 (Countable "commutator splitting envelopes")**.** Suppose P is a perfect group and $C \leq P$ a countable subgroup. Show that there is a countable subgroup $Q \leq P$ containing C such that for any element of $c \in C$, there exists an $n \in \mathbb{N}$ and elements $q_1, r_1, \ldots, q_n, r_n \in Q$ such that $c = [q_1, r_1] \cdots [q_n, r_n]$.

Exercise 3.2.14 (Closure properties of hypoabelian groups)**.** Show that:

1. Let $\varphi \colon G \to H$ be a group homomorphism and $P \le G$ a perfect subgroup. Then $\varphi(P) \le H$ is also a perfect subgroup.
2. Suppose we have an exact sequence of groups $A \to B \to C$ where A is abelian and C is hypoabelian. Then B is hypoabelian as well.
3. Let $\{G_i\}_{i \in I}$ be a set of hypoabelian groups. Then the product $\prod_i G_i$ is again hypoabelian.

Fact 3.2.15 (Hypoabelianisation of groups)*.* Every group G admits a maximal perfect subgroup P. Thus defining $G^{\mathrm{hypo}} \coloneqq G/P$ gives a left adjoint to the inclusion of the full subcategory of hypoabelian groups into all groups.

We learnt of the following lemma and its proof from Robert Burklund and Markus Land.

Lemma 3.2.16. *An ω_1–filtered colimit of hypoabelian groups is hypoabelian.*

Proof. Let $\{G_a\}_{a \in A}$ be an ω_1–filtered colimit of hypoabelian groups, $G \coloneqq \mathrm{colim}_{a \in A} G_a$, and suppose $P \le G$ is a perfect subgroup. We would like to argue that P is the trivial subgroup, and for this, we observe that it suffices to show that every countable perfect subgroup of P is trivial: this is because every element in a group belongs to a countable perfect subgroup (it is an elementary exercise to deduce this from Exercise 3.2.13). So let $C \le P$ be a countable perfect subgroup. Then since C is ω_1–compact, we see that the inclusion $C \le P \le G$ factors through some G_a. Hence, $C \le G_a$ may be viewed as a perfect subgroup of G_a, and so by hypothesis, it is trivial, as was to be shown. □

The ensuing construction is of fundamental importance in K–theory. It was first introduced by Kervaire, but it was rediscovered and extensively exploited by Quillen in his K–theoretic investigations, whence the name. To that end, we will need the following little exercise.

Exercise 3.2.17. Let $A \to B \leftarrow C \in \mathrm{An}$ and write $P \coloneqq A \times_B C$. Show that for every point $x \colon * \to P$, we get an associated fibre sequence

$$\Omega_x B \longrightarrow P \longrightarrow A \times C$$

of pointed anima.

Construction 3.2.18 (Quillen's plus–construction)**.** A direct functorial construction of Quillen's plus–construction can of course be given, but the approach taken here is purely from formal nonsense using the group–theoretic ingredients above. By Lemma 3.2.16, we know that the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ preserves ω_1–filtered colimits. Moreover, by Exercise 3.2.14 (3), the inclusion also preserves arbitrary products. We claim furthermore that the inclusion also preserves pullbacks. To wit, let

$A \to B \leftarrow C \in \mathrm{An}^{\mathrm{hypo}}$. By Exercise 3.2.17, for every point $x \in P := A \times_B C$, we get an associated fibration sequence which induces an exact sequence of groups

$$\pi_2(B, x) \longrightarrow \pi_1(P, x) \longrightarrow \pi_1(A \times C, x) \cong \pi_1(A, x) \times \pi_1(C, x)$$

Thus, by Exercise 3.2.14 (2, 3), we get that P is a hypoabelian anima as well, whence the preservation of pullbacks under the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ as claimed. All in all, by the Reflection Theorem 2.2.6, we get that $\mathrm{An}^{\mathrm{hypo}}$ admits a left adjoint which we write as $(-)^+ \colon \mathrm{An} \longrightarrow \mathrm{An}^{\mathrm{hypo}}$. This is what we shall refer to as Quillen's plus–construction.

One might object that the left adjoint $(-)^+$ was constructed purely abstractly and is "unusable/uncomputable". The following proposition will show that this is not the case and we can already deduce quite a few results purely from the existence of the left adjoint.

Proposition 3.2.19 (Omnibus plus–construction)**.**

(1) The adjunction unit $X \to X^+$ induces an equivalence $\mathbb{S}[X] \to \mathbb{S}[X^+]$ in Sp*,*

(2) The functor $(-)^+$ viewed as an endofunctor on An *preserves arbitrary coproducts,*

(3) The functor $(-)^+$ preserves finite products,

(4) Let $X \in \mathrm{An}$ and $x \in X$. Then the map $X \to X^+$ induces an isomorphism $\pi_1(X, x)^{\mathrm{hypo}} \xrightarrow{\cong} \pi_1(X^+, x)$ where the source is the hypoabelianisation of groups from Fact 3.2.15.

Proof. We leave (1) and (3) as an exercise to the reader. For (1), show by using the appropriate adjunctions that the map $\mathbb{S}[X] \to \mathbb{S}[X^+]$ induces an equivalence $\mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}[X^+], Y) \to \mathrm{Map}_{\mathrm{Sp}}(\mathbb{S}[X], Y)$ for all $Y \in \mathrm{Sp}$. For (3), use the adjunction $- \times Z \dashv \mathrm{Map}(Z, -)$ in An and argue by adjunctions.

Point (2), is an immediate consequence of the fact that left adjoints preserve all colimits and the immediate observation that the inclusion $\mathrm{An}^{\mathrm{hypo}} \subseteq \mathrm{An}$ is closed under coproducts.

Finally, to see (4), first note that since $(-)^+$ preserves coproducts by (2), it suffices to show this for connected anima X. We need to show that for any hypoabelian group G, the map $\pi_1 X \to \pi_1 X^+$ induces a bijection

$$\hom(\pi_1 X^+, G) \longrightarrow \hom(\pi_1 X, G)$$

of hom sets of groups. The strategy is to use the equivalence of categories

$$\Omega \colon \mathrm{An}_*^{\geq 1} \rightleftharpoons \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An}) : B \tag{3.6}$$

which is a standard result in higher algebra: here $\mathrm{An}_*^{\geq 1} \subseteq \mathrm{An}_*$ is the full subcategory of pointed connected anima, and $\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An})$ is the category of $\mathbb{E}_1$–groups in anima.

For the unique point $x\colon * \to X$, we may prolong it along $X \to X^+$ to view it also as the unique point of the connected anima X^+. Thus, we may view $X \to X^+$ as a map of pointed anima and so we obtain a map

$$\mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(X^+, BG) \longrightarrow \mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(X, BG). \tag{3.7}$$

This map is an equivalence since it can be re–expressed as the map of pullbacks

$$\mathrm{Map}_{\mathrm{An}}(X^+, BG) \times_{\mathrm{Map}_{\mathrm{An}}(*,BG)} \{1\} \longrightarrow \mathrm{Map}_{\mathrm{An}}(X, BG) \times_{\mathrm{Map}_{\mathrm{An}}(*,BG)} \{1\}$$

where $\mathrm{Map}_{\mathrm{An}}(X^+, BG) \to \mathrm{Map}_{\mathrm{An}}(X, BG)$ is an equivalence by definition of $(-)^+$ since G was hypoabelian. On the other hand, since BG is 1–truncated, the map (3.7) may be rewritten as

$$\begin{aligned} &\mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(B\pi_1 X^+, BG) \simeq \mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(\tau_{\leq 1} X^+, BG) \\ &\xrightarrow{\simeq} \mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(\tau_{\leq 1} X, BG) \simeq \mathrm{Map}_{\mathrm{An}_*^{\geq 1}}(B\pi_1 X, BG) \end{aligned}$$

Thus, by the categorical equivalence (3.6), the map (3.7) may further be identified with the equivalence

$$\hom(\pi_1 X^+, G) \xrightarrow{\simeq} \hom(\pi_1 X, G)$$

mapping sets, and so is a bijection of sets as was to be shown. □

As an immediate consequence of Proposition 3.2.19 (3), we see that:

Corollary 3.2.20. *The functor* $(-)^+\colon \mathrm{An} \to \mathrm{An}$ *induces a functor*

$$(-)^+\colon\ \mathrm{CMon}(\mathrm{An}) \longrightarrow \mathrm{CMon}(\mathrm{An}),$$

that is, the plus–construction of an $\mathbb{E}_\infty$*–monoid anima attains a canonical* $\mathbb{E}_\infty$*–monoid structure.*

We now use the plus–construction to provide a fix to the inability of Proposition 3.2.10 to completely compute the group–completion functor.

Theorem 3.2.21 ([Nik17, Thm. 9])**.** *There are equivalences*

$$M[M^{-1}]^+ \simeq M^+[M^{-1}] \simeq M^{\mathrm{gp}}$$

Proof. For the first equivalence, we need to show that the map

$$M[M^{-1}] \longrightarrow M^+[M^{-1}]$$

is a plus–construction. To this end, first note by Corollary 3.2.11 that $M^+[M^{-1}]$ has abelian fundamental group. Thus, to check the claim, it suffices to show that the map induces an equivalence upon applying $\mathrm{Map}_{\mathrm{An}}(-, Y)$ for an arbitrary $Y \in \mathrm{An}^{\mathrm{hypo}}$. As usual, we may assume for simplicity that the set of generators of $\pi_0 M$ consists of a single element $m \in \pi_0 M$. In this case, we just compute:

$$\begin{aligned}\mathrm{Map}(M^+[M^{-1}], Y) &\simeq \lim\Big(\cdots \mathrm{Map}(M^+, Y) \xrightarrow{m^*} \mathrm{Map}(M^+, Y) \xrightarrow{m^*} \mathrm{Map}(M^+, Y)\Big)\\ &\simeq \lim\Big(\cdots \mathrm{Map}(M, Y) \xrightarrow{m^*} \mathrm{Map}(M, Y) \xrightarrow{m^*} \mathrm{Map}(M, Y)\Big)\\ &\simeq \mathrm{Map}(M[M^{-1}], Y)\end{aligned}$$

as was to be shown.

Next, for the second equivalence, consider the commuting square

$$\begin{tikzcd} M[M^{-1}]^+ \arrow[r] \arrow[d] & M^{\mathrm{gp}} \arrow[d] \\ M^+[M^{-1}] \arrow[r] & (M^+)^{\mathrm{gp}} \end{tikzcd}$$

The left vertical map is an equivalence by the point above, the bottom horizontal is an equivalence by Corollary 3.2.11, and the right vertical is an equivalence by an immediate unwinding of universal properties. Thus, the top horizontal map is an equivalence, as desired. □

Example 3.2.22 (Localising the symmetric groups). Consider $\mathrm{Fin}^{\simeq} \simeq \coprod_{n\geq 0} B\Sigma_n \in \mathrm{CMon}(\mathrm{An})$ which is even a symmetric monoidal 1–category. As commutative monoids, we have $\pi_0\mathrm{Fin}^{\simeq} \cong \mathbb{N}$, and under this identification, it is clear (since everything is just a symmetric monoidal 1–category where it is easy to check) that the generator $1 \in \mathbb{N}$ corresponds to the component $B\Sigma_1 \simeq *$ and the addition map $B\Sigma_n \simeq B\Sigma_1 \times B\Sigma_n \to B\Sigma_{n+1}$ corresponds to the canonical inclusion group homomorphism $\Sigma_n \hookrightarrow \Sigma_{n+1}$. Therefore, the addition by 1 map $1+\colon \coprod_{n\geq 0} B\Sigma_n \to \coprod_{n\geq 0} B\Sigma_n$ looks like

$$\coprod_{n\geq 0} B\Sigma_n \xrightarrow{1+} \coprod_{n\geq 0} B\Sigma_n \xrightarrow{1+} \coprod_{n\geq 0} B\Sigma_n \xrightarrow{1+} \coprod_{n\geq 0} B\Sigma_n \xrightarrow{1+} \cdots$$

$$\begin{tikzcd} \vdots & \vdots & \vdots & \vdots & \cdots \\ B\Sigma_2 \arrow[ur] & B\Sigma_2 \arrow[ur] & B\Sigma_2 \arrow[ur] & B\Sigma_2 \arrow[ur] & \cdots \\ B\Sigma_1 \arrow[ur] & B\Sigma_1 \arrow[ur] & B\Sigma_1 \arrow[ur] & B\Sigma_1 \arrow[ur] & \cdots \\ B\Sigma_0 \arrow[ur] & B\Sigma_0 \arrow[ur] & B\Sigma_0 \arrow[ur] & B\Sigma_0 \arrow[ur] & \cdots \end{tikzcd}$$

and hence, we get that $\mathrm{Fin}^{\simeq}[(\mathrm{Fin}^{\simeq})^{-1}] \simeq \mathbb{Z} \times B\Sigma_\infty$. All in all, we see that

$$\mathrm{k}(\mathrm{Fin}) = (\mathrm{Fin}^{\simeq})^{\mathrm{gp}} \simeq \mathbb{Z} \times B\Sigma_\infty^+$$

In the preceding example, it was important that we use the plus–construction since we cannot apply Proposition 3.2.10 (4) because $\pi_1 B\Sigma_\infty \cong \Sigma_\infty$ is very nonabelian. As we shall see in Theorem 3.3.7, $\mathrm{k}(\mathrm{Fin})$ is in fact the sphere spectrum, and so it has many nontrivial higher homotopy groups in stark contrast to $\mathbb{Z} \times B\Sigma_\infty$ which is just a disjoint union of Eilenberg–Mac Lane spaces. However, there are interesting situations where we may dispense with the plus–construction, as the following example illustrates.

Example 3.2.23 (Localising unitary groups). Consider the topological category $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq}$ of finite–dimensional $\mathbb{C}$–vector spaces and isomorphisms. By the classical result that the group homomorphism $U(n) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ is a homotopy equivalence, we see that as an ∞–category, $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq}$ is equivalent to $\coprod_{n\geq 0} BU(n)$. This anima again has π_0 isomorphic to $\mathbb{N}$ as commutative monoids, and by the argument in Example 3.2.22, we see that

$$\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq}[(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq})^{-1}] \simeq \mathbb{Z} \times BU$$

But then $\pi_B U = 0$ and so we are in the situation of Proposition 3.2.10 to obtain that

$$\mathrm{ku} := \mathrm{k}(\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq}) \simeq \mathbb{Z} \times BU$$

Here ku is known as the *complex topological K–theory space* and is a very important object in higher algebra.

3.3 K–theory of finite sets

Our business in this subsection is to analyse $\mathrm{k}(\mathrm{Fin})$, culminating in a complete identification of it as the sphere spectrum in Theorem 3.3.7. While the proof, as we shall see, is more or less formal, the result is a deep one and it was first proved via extremely nontrivial computations. In some sense, it justifies that the sphere spectrum is the "correct" object to study since it says that the sphere spectrum is the output of universally group–completing the finite sets in the (higher) categorical setting.

Fact 3.3.1. We have a formula for the symmetric monoidal adjunction unit $\mathrm{Free}_{\mathbb{E}_\infty} \colon \mathrm{An} \to \mathrm{CMon}(\mathrm{An})$ from the Bousfield localisation $\mathrm{Pr}_L \rightleftharpoons \mathrm{Pr}_{L,\mathrm{sadd}}$ given as follows:

$$\mathrm{Free}_{\mathbb{E}_\infty}(X) \simeq \coprod_{n\geq 0} (X^{\times n})_{h\Sigma_n}$$

In particular, we see that $\mathrm{Free}_{\mathbb{E}_\infty}(*) \simeq \coprod_{n\geq 0} *_{h\Sigma_n} \simeq \coprod_{n\geq 0} B\Sigma_n \simeq \mathrm{Fin}^{\simeq}$ and since $* \in \mathrm{CAlg}(\mathrm{An}^{\times})$ is the tensor unit, we also get that $\mathrm{Fin}^{\simeq} \in \mathrm{CAlg}(\mathrm{CMon}(\mathrm{An})^{\otimes})$ is the tensor unit.

Fact 3.3.2. The structure $\mathrm{Fin}^{\simeq} \in \mathrm{CMon}(\mathrm{An})^{\otimes}$ is precisely given by the disjoint union of finite sets and the additional structure $\mathrm{Fin}^{\simeq} \in \mathrm{CAlg}(\mathrm{CMon}(\mathrm{An})^{\otimes})$ is given by the cartesian product of finite sets.

So far, we have viewed $\mathrm{Fin}^{\simeq}$ as an object in $\mathrm{CMon}(\mathrm{An})$ and to study $\mathrm{k}(\mathrm{Fin})$, we really want to view it as such. On the other hand, the group–completion K–theory functor from Definition 3.1.3 has domain $\mathrm{CMon}(\mathrm{Cat})$. We explain next how to relate these two points of view.

Observation 3.3.3. Let inclusion $\mathrm{An} \overset{i}{\hookrightarrow} \mathrm{Cat}$ from Fact 2.1.4 is a product–preserving left adjoint. Hence, as in Construction 3.1.5, we may quote [GGN15, Lem. 6.1] to get an identification of the morphism $i \otimes \mathrm{idAn} \otimes \mathrm{CMon} \to \mathrm{Cat} \otimes \mathrm{CMon}$ in Pr_L with the postcomposition morphism $\mathrm{CMon}(i)\colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CMon}(\mathrm{Cat})$. Thus, we get a commuting square

$$\begin{tikzcd}
\mathrm{An} \arrow[r, hook, "i"] \arrow[d, "\mathrm{Free}_{\mathbb{E}_\infty}"'] & \mathrm{Cat} \arrow[d, "\mathrm{Free}_{\mathbb{E}_\infty}"] \\
\mathrm{CMon}(\mathrm{An}) \arrow[r, hook, "\mathrm{CMon}(i)"'] & \mathrm{CMon}(\mathrm{Cat})
\end{tikzcd}$$

Moreover, the functor i is naturally symmetric monoidal with respect to the cartesian symmetric monoidal structures, and so all the functors in the square naturally refine to symmetric monoidal ones. In particular, the tensor unit $* \in \mathrm{An}$ is sent to the tensor unit $\mathrm{Fin}^{\simeq} = \mathrm{Free}_{\mathbb{E}_\infty}(*)$ which may be viewed to live either in $\mathrm{CMon}(\mathrm{An})$ or $\mathrm{CMon}(\mathrm{Cat})$. Furthermore, under the Bousfield colocalisation $\mathrm{CMon}(i)\colon \mathrm{CMon}(\mathrm{An}) \rightleftharpoons \mathrm{CMon}(\mathrm{Cat}) : \mathrm{CMon}((-)^{\simeq})$, we see that the construction $\mathrm{Fin} \in \mathrm{CMon}(\mathrm{Cat}) \mapsto \mathrm{k}(\mathrm{Fin}) \in \mathrm{CGrp}$ factors through $\mathrm{Fin}^{\simeq} \in \mathrm{CMon}(\mathrm{An})$. Thus, as far as studying $\mathrm{k}(\mathrm{Fin})$ is concerned, we are free to view $\mathrm{Fin}^{\simeq}$ as an object in $\mathrm{CMon}(\mathrm{An})$.

Notation 3.3.4. Recall from Theorem 2.5.15 that there is an adjunction $\mathrm{CGrp} \rightleftharpoons \mathrm{Sp}$ coming from the Bousfield localisation $\mathrm{Pr}_{L,\mathrm{add}} \rightleftharpoons \mathrm{Pr}_{L,\mathrm{st}}$. This is an important adjunction and so we will dignify it with the following names:

$$\begin{tikzcd}
\mathrm{CGrp} \arrow[r, shift left, "B^\infty"] & \mathrm{Sp} \arrow[l, shift left, "\Omega^\infty"]
\end{tikzcd}$$

Furthermore, we write $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ for the full subcategory of *connective spectra*, i.e. those spectra whose homotopy groups are concentrated in degrees ≥ 0 (note that a spectrum can have homotopy groups in negative degrees! We will study this phenomenon in more detail later when we meet t–structures).

A very important fundamental theorem in higher algebra is the following:

Theorem 3.3.5. *The functor* $B^\infty\colon \mathrm{CGrp} \to \mathrm{Sp}$ *factors through* $\mathrm{Sp}_{\geq 0}$ *and it induces an equivalence* $B^\infty\colon \mathrm{CGrp} \xrightarrow{\simeq} \mathrm{Sp}_{\geq 0}$. *Consequently, we have a symmetric monoidal equivalence* $B^\infty\colon \mathrm{CGrp}^{\otimes} \xrightarrow{\simeq} \mathrm{Sp}^{\otimes}_{\geq 0}$.

Remark 3.3.6. In particular, since $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ factors as $\mathbb{S}[-]\colon \mathrm{An} \xrightarrow{\mathrm{Free}^{\mathrm{gp}}_{\mathbb{E}_\infty}} \mathrm{CGrp} \xrightarrow[\simeq]{B^\infty} \mathrm{Sp}$, the functor $\mathbb{S}[-]$ also lands in $\mathrm{Sp}_{\geq 0}$.

Theorem 3.3.7 (Barratt–Priddy–Quillen, Segal)**.** *There are equivalences* $\Omega^\infty \mathbb{S} \simeq \mathrm{k}(\mathrm{Fin}) \simeq \mathbb{Z} \times B\Sigma_\infty^+$*. In fact, the first equivalence is even an equivalence in* $\mathrm{CAlg}(\mathrm{CGrp}^\otimes)$*.*

Proof. The second equivalence is supplied precisely by Example 3.2.22. To see that we have the first equivalence in $\mathrm{CAlg}(\mathrm{CGrp}^\otimes)$, first note that by Theorem 3.3.5, we obtain an equivalence $B^\infty \colon \mathrm{CAlg}(\mathrm{CGrp}^\otimes) \simeq \mathrm{CAlg}(\mathrm{Sp}^\otimes) : \Omega^\infty$. Next, observe that $\mathrm{k}(\mathrm{Fin}) \coloneqq (\mathrm{Fin}^\simeq)^{\mathrm{gp}} \simeq \mathrm{Free}_{\mathbb{E}_\infty}(*)^{\mathrm{gp}}$ where the second equivalence is by Fact 3.3.1. Now by Theorem 2.5.15, we have a symmetric monoidal functor

$$\mathrm{An} \xrightarrow{\mathrm{Free}_{\mathbb{E}_\infty}} \mathrm{CMon} \xrightarrow{(-)^{\mathrm{gp}}} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{B^\infty} \mathrm{Sp}$$

coming from the appropriate adjunction units. In particular, it sends the tensor unit $*$ in $\mathrm{An}^\times$ to the tensor unit $\mathbb{S}$ in $\mathrm{Sp}^\otimes$. Thus, we get the equivalence $B^\infty \mathrm{k}(\mathrm{Fin}) \simeq \mathbb{S}$ in $\mathrm{CAlg}(\mathrm{Sp}^\otimes)$, or equivalently,

$$\mathrm{k}(\mathrm{Fin}) \simeq \Omega^\infty \mathbb{S}$$

in $\mathrm{CAlg}(\mathrm{CGrp}^\otimes)$ as was to be shown. □

4 Algebraic K–theory: additivity and localisation

4.1 K-theory of stable categories and the Q-construction

In this section we will discuss algebraic K-theory of stable ∞-categories.

Notation 4.1.1. Let $R \in \mathrm{CAlg}(\mathrm{Sp}^{\otimes})$. Let Perf_R denote the smallest thick subcategory of Mod_R containing R. Recall that this means that Perf_R is the smallest stable subcategory of Mod_R which is closed under retracts and contains R.

Example 4.1.2. Let $R \in \mathrm{CRing}$. Every finitely generated projective R-module is a retract of a finitely generated free R-module, so $\mathrm{Proj}_R \subseteq \mathrm{Perf}_R$.

The following exercise provides a convenient characterisation of perfect R-modules. The reader can also consult [Lur17, §7.2.4].

Exercise 4.1.3. Let $R \in \mathrm{CAlg}(\mathrm{Sp}^{\otimes})$ and let $P \in \mathrm{Mod}_R$. Show that P is perfect if and only if P is a compact object of Mod_R if and only if P is a dualisable object of Mod_R.

Fact 4.1.4. If $R \in \mathrm{CRing}$, then there is an equivalence of ∞-categories

$$\mathrm{Perf}_R \simeq \mathcal{D}^{\mathrm{perf}}(R),$$

where the latter denotes the perfect derived ∞-category of R. An object of $\mathcal{D}^{\mathrm{perf}}(R)$ can be represented by a bounded complex of finitely generated projective R-modules.

The discussion in Fact 4.1.4 provides an indication of why we might expect algebraic K-theory to have functoriality in perfect modules rather than just projective modules. We recommend to work through the following exercise:

Exercise 4.1.5. Let $R \in \mathrm{CRing}$. Show that the assignment

$$\pi_0(\mathcal{D}^{\mathrm{perf}}(R)^{\simeq}) \to \mathrm{k}_0(R)$$

defined $P \mapsto \sum_i (-1)^i [P_i]$ exhibits $\mathrm{k}_0(R)$ as the quotient of $\pi_0(\mathcal{D}^{\mathrm{perf}}(R)^{\simeq})$ by the equivalence relation $\sim$ generated by $y \sim x \oplus z$ for every fiber sequence $x \to y \to z$ in $\mathcal{D}^{\mathrm{perf}}(R)$.

This motivates the following defintion which works in general for a stable ∞-category.

Definition 4.1.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ and define

$$\mathcal{K}_0(\mathcal{C}) = \pi_0(\mathcal{C}^{\simeq})/\sim,$$

where $\sim$ is the equivalence relation generated by $y \sim x \oplus z$ for every fiber sequence $x \to y \to z$ in $\mathcal{C}$.

In the definition above, we see that taking direct sums in $\mathcal{C}$ endows $\mathcal{K}_0(\mathcal{C})$ with the structure of an abelian group with inverses given by $-[x] = [\Sigma x]$. Intuitively, we no longer need to group complete on π_0 but we rather have to split extensions. Our next goal will be to construct the entire K-theory anima as a functor $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

The Q-construction

One can construct the functor $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ using Waldhausen's S-construction or Quillen's Q-construction. The crucial difference between the two is that the former is built out of the arrow category while the later is built from the twisted arrow category. For the purposes of this course it will be convenient to employ Quillen's Q-construction in our construction of $\mathcal{K}$. Let us begin by giving a informal discussion of the twisted arrow category.

Idea 4.1.7. Let $\mathcal{C} \in \mathrm{Cat}$.

1. The left twisted arrow category of $\mathcal{C}$ is the ∞-category $\mathrm{TwAr}^l(\mathcal{C})$ whose objects are morphisms of $\mathcal{C}$. A morphism from $x \to y$ to $x' \to y'$ is the datum of a commutative diagram
$$\begin{tikzcd} x \arrow[d] & x' \arrow[l] \arrow[d] \\ y \arrow[r] & y' \end{tikzcd}$$
2. The right twisted arrow category of $\mathcal{C}$ is the ∞-category $\mathrm{TwAr}^l(\mathcal{C})$ whose objects are morphisms of $\mathcal{C}$. A morphism from $x \to y$ to $x' \to y'$ is the datum of a commutative diagram
$$\begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d] \\ y & y' \arrow[l] \end{tikzcd}$$

Definition 4.1.8. The left twisted arrow ∞-category of $\mathcal{C}$ is defined by the pullback

$$\begin{tikzcd} \mathrm{TwAr}^l(\mathcal{C}) \arrow[r] \arrow[d] & */\mathrm{An} \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{Map}_{\mathcal{C}}"] & \mathrm{An} \end{tikzcd}$$

in Cat. Define $\mathrm{TwAr}^r(\mathcal{C}) = \mathrm{TwAr}^l(\mathcal{C})^{\mathrm{op}}$.

Warning 4.1.9. Equivalently, we have that $\mathrm{TwAr}^l(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is the left fibration obtained by the unstraightening of the mapping anima functor $\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$. As a consequence, our presentation here is slightly circular since the construction of the mapping anima functor $\mathrm{Map}_{\mathcal{C}}$ is most commonly done by explicitly writing down a left fibration $\mathrm{TwAr}^l(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and then forming the straightening of this left fibration. Indeed, the explicit definition is given by

$$\mathrm{TwAr}^l(\mathcal{C}) = \mathrm{Hom}_{\mathrm{sSet}}((\Delta^n)^{\mathrm{op}} \star \Delta^n, \mathcal{C})$$

and we refer the reader to [Lur17, §5.2.1] for a detailed discussion of this perspective. These subleties will not be important for our presentation.

Exercise 4.1.10. Show that the objects and morphisms of $\mathrm{TwAr}^l(\mathcal{C})$ and $\mathrm{TwAr}^r(\mathcal{C})$ indeed are given as in Idea 4.1.7. **Hint:** *use that the mapping anima of a pullback is the pullback of the mapping anima to obtain the description of morphisms.*

Exercise 4.1.11. In this exercise we let $\mathcal{C} = [n]$ for some $n \geq 0$.

1. Show that $\mathrm{TwAr}^r([0]) = 0$.
2. Show that $\mathrm{TwAr}^r([1])$ can be depicted by a span

$$\begin{tikzcd}
& (0 \leq 1) \arrow[dl] \arrow[dr] & \\
(0 \leq 0) & & (1 \leq 1)
\end{tikzcd}$$

3. Show that $\mathrm{TwAr}^r([2])$ can be depicted by

$$\begin{tikzcd}
& & (0 \leq 2) \arrow[dl] \arrow[dr] & & \\
& (0 \leq 1) \arrow[dl] \arrow[dr] & & (1 \leq 2) \arrow[dl] \arrow[dr] & \\
(0 \leq 0) & & (1 \leq 1) & & (2 \leq 2)
\end{tikzcd}$$

With these pictures in mind, we have the following definition:

Definition 4.1.12. Let $\mathcal{C} \in \mathrm{Cat}$ and assume that $\mathcal{C}$ admits pullbacks. For $n \geq 0$, let

$$\mathrm{Q}_n(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{TwAr}^r([n]), \mathcal{C})$$

denote the full subcategory spanned by those functors which take every square

$$\begin{tikzcd}
& (i \leq l) \arrow[dl] \arrow[dr] & \\
(i \leq k) \arrow[dr] & & (j \leq l) \arrow[dl] \\
& (j \leq k) &
\end{tikzcd}$$

to a pullback square in $\mathcal{C}$.

Exercise 4.1.13. Show that $\mathcal{C} \mapsto \mathrm{Q}(\mathcal{C})$ determines a functor $\mathrm{Q}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{sCat}^{\mathrm{st}}$.

We see that $\mathrm{Q}_0(\mathcal{C}) \simeq \mathcal{C}$. Similarly, an object of $\mathrm{Q}_1(\mathcal{C})$ is given by a span

$$\begin{tikzcd}
& x_{01} \arrow[dl] \arrow[dr] & \\
x_{00} & & x_{11}
\end{tikzcd}$$

in $\mathcal{C}$. Finally, we see that an object of $\mathrm{Q}_2(\mathcal{C})$ is given by a diaram

$$\begin{tikzcd}
& & x_{02} \arrow[dl] \arrow[dr] & & \\
& x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & \\
x_{00} & & x_{11} & & x_{22}
\end{tikzcd}$$

in $\mathcal{C}$, where the top square is a pullback.

The following exercise provides a good way of getting used to the definitions. We will use it later, so the reader is encouraged to give it a try.

Exercise 4.1.14. For $n \geq 0$, let $\mathcal{I}_n$ denote the subposet of $\mathrm{TwAr}^r([n])$ spanned by those 0-simplices $(i \leq j)$ which satisfy that $j \leq i+1$. We depict $\mathcal{I}_n$ by

$$\begin{tikzcd}
& (0 \leq 1) \arrow[dl] \arrow[dr] & & (1 \leq 2) \arrow[dl] \arrow[dr] & & (n-1 \leq n) \arrow[dl] \arrow[dr] & \\
(0 \leq 0) & & (1 \leq 1) & & \cdots & & (n \leq n)
\end{tikzcd}$$

Show that a functor $F\colon \mathrm{TwAr}^r([n]) \to \mathcal{C}$ lies in $\mathrm{Q}_n(\mathcal{C})$ if and only if F is right Kan extended from the restriction $\mathcal{I}_n \to \mathrm{TwAr}^r([n]) \to \mathcal{C}$. Conclude that the restriction functor furnishes an equivalence $\mathrm{Q}_n(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{I}_n, \mathcal{C})$ for every $n \geq 0$.

Proposition 4.1.15 (Barwick)**.** *The simplicial ∞-category $\mathrm{Q}(\mathcal{C})$ is a Segal object for every ∞-category $\mathcal{C}$ which admits pullbacks. This means that the Segal maps*

$$\mathrm{seg}_i\colon [1] \to [n], 0, 1 \mapsto i-1, i$$

for $1 \leq i \leq n$, induce an equivalence of ∞-categories

$$\mathrm{Q}_n(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C}) \times_{\mathrm{Q}_0(\mathcal{C})} \mathrm{Q}_1(\mathcal{C}) \times_{\mathrm{Q}_0(\mathcal{C})} \cdots \times_{\mathrm{Q}_0(\mathcal{C})} \mathrm{Q}_1(\mathcal{C})$$

for every $n \geq 1$.

Proof. The Segal maps induce an equivalence $\mathcal{J}_n \simeq \mathcal{J}_1 \sqcup_{\mathcal{J}_0} \cdots \sqcup_{\mathcal{J}_0} \mathcal{J}_1$ in Cat, so

$$\begin{aligned}
\mathrm{Q}_n(\mathcal{C}) &\simeq \mathrm{Fun}(\mathcal{J}_1 \sqcup_{\mathcal{J}_0} \cdots \sqcup_{\mathcal{J}_0} \mathcal{J}_1, \mathcal{C}) \\
&\simeq \mathrm{Fun}(\mathcal{J}_1, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{J}_0,\mathcal{C})} \cdots \times_{\mathrm{Fun}(\mathcal{J}_0,\mathcal{C})} \mathrm{Fun}(\mathcal{J}_1, \mathcal{C}) \\
&\simeq \mathrm{Q}_1(\mathcal{C}) \times_{\mathrm{Q}_0(\mathcal{C})} \cdots \times_{\mathrm{Q01}(\mathcal{C})} \mathrm{Q}_1(\mathcal{C}),
\end{aligned}$$

where we have used that $\mathrm{Q}_n(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{I}_n, \mathcal{C})$ from Exercise 4.1.14. $\square$

Remark 4.1.16. In the setting of Proposition 4.1.15, the Segal object $\mathrm{Q}(\mathcal{C})$ in sCat is also complete in the sense that the square

$$\begin{tikzcd}
\mathrm{Q}_0(\mathcal{C}) \arrow[r, "s"] \arrow[d, "\Delta"] & \mathrm{Q}_3(\mathcal{C}) \arrow[d, "{(d_{02},d_{13})}"] \\
\mathrm{Q}_0(\mathcal{C})^2 \arrow[r, "{(s,s)}"] & \mathrm{Q}_1(\mathcal{C})^2
\end{tikzcd}$$

is a pullback in Cat, where $d_{02}\colon [1] \to [3]$ is the unique map that misses 0 and 2 and similarly for d_{13}.

Construction 4.1.17. Recall that the Yoneda embedding induces an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{sAn}, \mathrm{Cat}) \xrightarrow{y^*} \mathrm{Fun}(\Delta, \mathrm{Cat})$$

as discussed in Theorem 2.1.25. Let ac denote the essentially unique colimit preserving functor $\mathrm{sAn} \to \mathrm{Cat}$ such that the composite

$$\Delta \xrightarrow{y} \mathrm{sAn} \xrightarrow{\mathrm{ac}} \mathrm{Cat}$$

is canonically equivalent to the functor $[-]\colon \Delta \to \mathrm{Cat}$ determined by $[n] \mapsto [n]$. Equivalently, the functor ac is obtained as the left Kan extension in the following diagram

$$\begin{tikzcd}
\Delta \arrow[r, "{[-]}"] \arrow[d, "y"] & \mathrm{Cat} \\
\mathrm{sAn} \arrow[ur, dashed, "\mathrm{ac}"'] &
\end{tikzcd}$$

We refer to ac as the associated category functor. The functor $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}$ admits a right adjoint $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sAn}$ determined by the construction $\mathcal{C} \mapsto \mathrm{Map}_{\mathrm{Cat}}([-], \mathcal{C})$. We refer to N as the Rezk nerve.

Exercise 4.1.18. Let $X \in \mathrm{sAn}$. Show that $|\mathrm{ac}(X)| \simeq |X|$. **Hint:** *observe that the functor* $\mathrm{An} \subseteq \mathrm{Cat} \xrightarrow{\mathrm{N}} \mathrm{sAn}$ *is constant and compare adjoints.*

Remark 4.1.19. In fact, the Rezk nerve $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sAn}$ is fully faithful with essential image spanned by the complete Segal anima. This is a difficult result established by Joyal–Tierney, Lurie, and Rezk and will not play an important role for us.

The associated category functor $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}$ does not preserve pullbacks. However, we record the following special situation where it does.

Lemma 4.1.20. *Let $X \to Y \leftarrow Z$ be maps of Segal anima. The canonical functor*

$$\mathrm{ac}(X \times_Y Z) \to \mathrm{ac}(X) \times_{\mathrm{ac}(Y)} \mathrm{ac}(Z)$$

is fully faithful, and essentially surjective if $\mathrm{ac}(X) \to \mathrm{ac}(Y)$ *is a bicartesian fibration.*

Proof. See step (4) in proof of [HW21, Theorem IV.18]. ☐

Definition 4.1.21. Let $F \in \mathrm{Fun}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$ and define $\mathrm{Span}^F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}$ by

$$\mathrm{Span}^F \colon \mathrm{Cat}^{\mathrm{st}} \xrightarrow{\mathrm{Q}} \mathrm{sCat}^{\mathrm{st}} \xrightarrow{F} \mathrm{sAn} \xrightarrow{\mathrm{ac}} \mathrm{Cat}.$$

Notation 4.1.22. If $F = (-)^{\simeq} \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$, then we write $\mathrm{Span} = \mathrm{Span}^{(-)^{\simeq}}$.

Remark 4.1.23. In the following, we let $F \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be a functor.

1. Using Exercise 4.1.18, we conclude that $|\mathrm{Span}^F(\mathcal{C})| = |\mathrm{ac}F\mathrm{Q}(\mathcal{C})| \simeq |F\mathrm{Q}(\mathcal{C})|$.
2. Recall that $\mathrm{Q}(\mathcal{C})$ is a Segal object by virtue of Proposition 4.1.15. Consequently, if F preserves pullbacks, then $F\mathrm{Q} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{sAn}$ is a Segal anima. Later we will see that it suffices to assume that F is additive for $F\mathrm{Q}(\mathcal{C})$ to be a Segal anima. We note $\mathrm{Q}(\mathcal{C})^{\simeq}$ is a Segal anima since the core functor $F = (-)^{\simeq}$ preserves limits.

Definition 4.1.24 (K-theory of stable ∞-categories)**.** Define $\mathcal{K} \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ by

$$\mathcal{K} \colon \mathrm{Cat}^{\mathrm{st}} \xrightarrow{\mathrm{Span}} \mathrm{Cat} \xrightarrow{|-|} \mathrm{An} \xrightarrow{\Omega} \mathrm{An},$$

where the loop space is formed at the base object $0 \in \mathcal{C}^{\simeq} \simeq \mathrm{Span}(\mathcal{C})^{\simeq}$.

Construction 4.1.25. The construction $R \mapsto \mathrm{Perf}_R$ determines a functor

$$\mathrm{CAlg}(\mathrm{Sp}^{\otimes}) \to \mathrm{Cat}^{\mathrm{st}}$$

sending $R \to S$ in $\mathrm{CAlg}(\mathrm{Sp}^{\otimes})$ to the exact functor $\mathrm{Perf}_R \to \mathrm{Perf}_S$ given by $P \mapsto P \otimes_R S$.

Definition 4.1.26 (K-theory of $\mathbb{E}_\infty$-rings)**.** Define $\mathcal{K} \colon \mathrm{CAlg}(\mathrm{Sp}^{\otimes}) \to \mathrm{An}$ by

$$\mathcal{K} \colon \mathrm{CAlg}(\mathrm{Sp}^{\otimes}) \xrightarrow{\mathrm{Perf}} \mathrm{Cat}^{\mathrm{st}} \xrightarrow{\mathcal{K}} \mathrm{An}$$

Construction 4.1.27. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ and consider the functor $\mathcal{C} \to \mathrm{Q}_1(\mathcal{C})$ defined by $x \mapsto (0 \leftarrow x \to 0)$. This induces a functor of simplicial ∞-categories

$$\mathrm{const}_1\mathcal{C} \to \mathrm{Q}(\mathcal{C}),$$

where the former denotes the constant simplicial ∞-category with $\mathcal{C}$ in simplicial degree 1. By applying the realisation functor to this map we obtain a map of anima

$$\mathcal{C}^{\simeq} \to \mathcal{K}(\mathcal{C})$$

which is checked to be natural in $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$.

Fact 4.1.28. Recall from Construction 4.1.27 that we have a natural transformation

$$(-)^{\simeq} \Rightarrow \mathcal{K}$$

of functors $\mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$. If $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$, then the induced map

$$\pi_0(\mathcal{C}^{\simeq}) \to \pi_0\mathcal{K}(\mathcal{C})$$

exhibits the target as the quotient of the source by the equivalence relation $\sim$ generated by $y \sim x \oplus z$ for every fiber sequence $x \to y \to z$ in $\mathcal{C}$. This is most naturally proved using Waldhausen's S-construction and we refer the reader to [CDH+20, Appendix B.1] or [HW21]. As a consequence, our definition of $\mathcal{K}$ using the Q-construction agrees with Definition 4.1.6 on π_0.

4.2 Verdier sequences

We discuss the notion of (split) Verdier sequences following [CDH+20, Appendix A].

Verdier sequences

Definition 4.2.1. A sequence of stable ∞-categories and exact functors

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

with vanishing composite is a *Verdier sequence* if it is both a fiber and a cofiber sequence in $\mathrm{Cat}^{\mathrm{st}}_{\infty}$. We say that i is a Verdier inclusion and that p is a Verdier projection.

It will be essential to understand how fibers and cofibers are calculated in $\mathrm{Cat}^{\mathrm{st}}_{\infty}$.

Observation 4.2.2. Let $p\colon \mathcal{D} \to \mathcal{E}$ be in $\mathrm{Cat}^{\mathrm{st}}_{\infty}$. The inclusion $\mathrm{Cat}^{\mathrm{st}} \hookrightarrow \mathrm{Cat}$ preserves limits, the fiber of p in $\mathrm{Cat}^{\mathrm{st}}$ is the full subcategory

$$\mathrm{fib}(p) = \{d \in \mathcal{D} \mid p(d) \simeq 0\}.$$

This is in fact a stable subcategory since p is exact and it is straightforward to verify that $\mathrm{fib}(p) \subseteq \mathcal{D}$ satisfies the universal property. To summarize, if $p\colon \mathcal{D} \to \mathcal{E}$ is an exact functor of stable ∞-categories, then $\mathrm{fib}(p) \hookrightarrow \mathcal{D} \to \mathcal{E}$ is a fiber sequence in $\mathrm{Cat}^{\mathrm{st}}$. Note that $\mathrm{fib}(p)$ is a thick subcategory of $\mathcal{D}$.

Cofibers in $\mathrm{Cat}^{\mathrm{st}}$ are more interesting as these are given by Verdier quotients.

Definition 4.2.3. Let $i\colon \mathcal{C} \to \mathcal{D}$ denote a functor in $\mathrm{Cat}^{\mathrm{st}}$.

1. A morphism in $\mathcal{D}$ is an equivalence modulo $\mathcal{C}$ if its fiber lies in the smallest stable subcategory spanned by the essential image of $i\colon \mathcal{C} \to \mathcal{D}$.
2. Let $\mathcal{D}/\mathcal{C}$ denote the Dwyer–Kan localisation of $\mathcal{D}$ with respect to the equivalences modulo $\mathcal{C}$.

We have the following crucial result (cf. [NS18, Theorem I.3.3]).

Proposition 4.2.4. *Let $i\colon \mathcal{C} \to \mathcal{D}$ be a functor in* $\mathrm{Cat}^{\mathrm{st}}$*. Then:*

1. *The ∞-category $\mathcal{D}/\mathcal{C}$ is stable and the localisation functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.*
2. *For every stable ∞-category $\mathcal{E}$, the restriction functor*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$

is fully faithful and its essential image is spanned by those functors which vanish after precomposition with i. In particular, the sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is a cofiber sequence in $\mathrm{Cat}^{\mathrm{st}}$.

3. *For all $x, y \in \mathcal{D}$ with image $\bar{x}, \bar{y} \in \mathcal{D}/\mathcal{C}$, the mapping space in $\mathcal{D}/\mathcal{C}$ is given by*

$$\mathrm{Map}_{\mathcal{D}/\mathcal{C}}(\bar{x}, \bar{y}) \simeq \mathrm{colim}_{z \in \mathcal{C}_{/y}} \mathrm{Map}_{\mathcal{D}}(x, \mathrm{cofib}(z \to y)),$$

where the colimit is filtered.

Exercise 4.2.5. Show that $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a cofiber sequence using (2) above.

Exercise 4.2.6. Consider the cofiber sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \to \mathcal{D}/\mathcal{C}$ in $\mathrm{Cat}^{\mathrm{st}}$. Show that the sequence is a fiber sequence if and only if for every stable ∞-category $\mathcal{E}$, the functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{E}, \mathcal{D})$$

given by postcomposition with i is fully faithful with essential image spanned by those functors whose postcomposition with the localisation functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ vanishes.

It follows from Exercise 4.2.6 that the sequence

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is a fiber sequence precisely if the functor $i\colon \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as the fiber of $\mathcal{D} \to \mathcal{D}/\mathcal{C}$, in other words that $\mathcal{C} \simeq \mathrm{fib}(\mathcal{D} \to \mathcal{D}/\mathcal{C})$. In particular, we note that $\mathcal{C}$ needs to be a thick subcategory. The following result ensures that this is in fact sufficient to guarantee that our original sequence is a fiber sequence and thus a Verdier sequence as well.

Lemma 4.2.7. *The fiber of the localisation functor $p\colon \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is the full subcategory of $\mathcal{D}$ spanned by those objects which are retracts of objects in $\mathcal{C}$.*

Proof. Any retract of an object of $\mathcal{C}$ lies in the fiber of p. Conversely, suppose that $x \in \mathrm{fib}(p)$. We want to show that x is a retract of an object of $\mathcal{C}$. By (2) of Proposition 4.2.4, note that every exact functor $f\colon \mathcal{D} \to \mathrm{Sp}$ that sends $\mathcal{C}$ to zero factors over $p\colon \mathcal{D} \to \mathcal{D}/\mathcal{C}$ and hence vanishes on $\mathrm{fib}(p)$. Consider the exact functor $\varphi_x\colon \mathcal{D} \to \mathrm{Sp}$ given by

$$\varphi_x(y) = \mathrm{colim}_{(\beta\colon z \to y) \in \mathcal{C}_{/y}} \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(\beta)),$$

where $\mathcal{C}_{/y} = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/y}$. We claim that φ_x vanishes on $\mathcal{C}$. Indeed, for any $y \in \mathcal{C}$, we have that $\mathcal{C}_{/y}$ has a final object given by $\mathrm{id}\colon y \to y$, so

$$\varphi_x(y) \simeq \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(\mathrm{id}_y)) \simeq 0.$$

It follows that φ_x vanishes on $\mathrm{fib}(p)$ which in particular means that

$$0 \simeq \varphi_x(x) = \mathrm{colim}_{(\beta\colon z \to x) \in \mathcal{C}_{/x}} \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(\beta)).$$

For $0 \to x$, the class of $\mathrm{id}_x \in \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(0 \to x))$ must vanish in the colimit, so there is a morphism $\beta\colon z \to x$ in $\mathcal{C}$ such that id_x is in the kernel of

$$\mathrm{map}_{\mathcal{D}}(x, x) \to \mathrm{map}_{\mathcal{D}}(x, \mathrm{cofib}(\beta)).$$

Since $\mathcal{D}$ is stable, this means that $\mathrm{id}_x\colon x \to x$ factors over $\beta\colon z \to x$ which exhibits x as a retract of z. □

Corollary 4.2.8. *If $\mathcal{C}$ is a thick subcategory of a stable ∞-category $\mathcal{D}$, then*

$$\mathcal{C} \hookrightarrow \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is a Verdier sequence.

We obtain the following omnibus result for Verdier sequences:

Proposition 4.2.9 (Omnibus Verdier sequence)**.** *Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in* $\mathrm{Cat}^{\mathrm{st}}$ *with vanishing composite. The following are equivalent:*

1. *The sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a Verdier sequence.*
2. *(i) The functor i is fully faithful with essential image closed under retracts.*
   (ii) The functor p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.
3. *(i) The functor p is a Dwyer–Kan localisation.*
   (ii) The functor i exhibits $\mathcal{C}$ as the fiber of p.

Proof. We first prove that (1) ⇒ (2). Suppose that $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a Verdier sequence. In particular, it is a cofiber sequence which proves 2.(*ii*) by Proposition 4.2.4. It is also a fiber sequence, so the essential image of i is closed under retracts by Lemma 4.2.7 which proves 2.(*i*). The direction (2) ⇒ (1) follows again from Proposition 4.2.4 and Lemma 4.2.7. We leave (1) ⇔ (3) as an exercise. □

Corollary 4.2.10. *Every Dwyer–Kan localisation is essentially surjective. In particular, every Verdier projection is essentially surjective.*

Proof. Let $p\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ be a Dwyer–Kan localisation and let $\mathcal{D}$ denote the essential image of p. We wish to prove that $\mathcal{D} \hookrightarrow \mathcal{C}[W^{-1}]$ is an equivalence. Note that $p\colon \mathcal{C} \to \mathcal{D}$ induces a functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{W^{-1}}(\mathcal{C}, \mathcal{E})$$

which is an equivalence for every $\mathcal{E} \in \mathrm{Cat}$. Indeed, given a functor $\mathcal{C} \to \mathcal{E}$ which carries W to equivalence, we obtain a functor $\mathcal{D} \hookrightarrow \mathcal{C}[W^{-1}] \to \mathcal{E}$ by the universal property of p. This proves that $\mathcal{C} \to \mathcal{D}$ satisfies the same universal property as $\mathcal{C} \to \mathcal{C}[W^{-1}]$ which implies that $\mathcal{D} \hookrightarrow \mathcal{C}[W^{-1}]$ is an equivalence. ☐

Lemma 4.2.11. *Verdier projections are stable under pullback. More precisely, if*

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "k"] \arrow[d, "p'"] & \mathcal{D} \arrow[d, "p"] \\
\mathcal{E}' \arrow[r, "l"] & \mathcal{E}
\end{tikzcd}$$

is a pullback in $\mathrm{Cat}_\infty^{\mathrm{st}}$ *in which p is a Verdier projection, then p' is a Verdier projection.*

Proof. See [CDH+20, Lemma A.1.11]. ☐

Split Verdier sequences

Definition 4.2.12. A Verdier sequence

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

is *split* if p admits both a left and a right adjoint. We say that i is a split Verdier inclusion and that p is a split Verdier projection.

We obtain the following characterisation of split Verdier sequences.

Proposition 4.2.13. *Let $p\colon \mathcal{D} \to \mathcal{E}$ and $i\colon \mathcal{C} \to \mathcal{D}$ be functors in* $\mathrm{Cat}^{\mathrm{st}}$.

1. *The functor p is a split Verdier projection if and only if it admits fully faithful left and right adjoints.*
2. *The functor i is a split Verdier inclusion if and only if it is fully faithful and admits left and right adjoints.*

Proof. See [CDH+20, Corollary A.2.6]. ☐

Remark 4.2.14. Suppose that $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a split Verdier sequence. It follows from Proposition 4.2.13 that we obtain the following diagram of adjunctions

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] & \mathcal{D} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "p"] & \mathcal{E} \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

This is called a *stable recollement*.

Corollary 4.2.15. *A pullback of a split Verdier projection is a split Verdier projection.*

Proof. Consider a pullback square in $\mathrm{Cat}^{\mathrm{st}}$

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "k'"] \arrow[d, "p'"] & \mathcal{D} \arrow[d, "p"] \\
\mathcal{E}' \arrow[r, "k"] & \mathcal{E}
\end{tikzcd}$$

where p is a split Verdier projection with fully faithful left adjoint l and right adjoint r (which exist by Proposition 4.2.13). We have to exhibit fully faithful left and right adjoints of the pullback p'. Using the universal property of pullbacks and that both l and r are fully faithful we obtain the desired left and right adjoints of p'. Using the formula for mapping anima in a pullback, we see that those functors are fully faithful. □

Exercise 4.2.16 (Waldhausen's foundational split Verdier sequence)**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. Consider the sequence

$$\mathcal{C} \xrightarrow{r} \mathrm{Ar}(\mathcal{C}) \xrightarrow{t} \mathcal{C},$$

where $r(x) = (x \to 0)$ and $t(x \to y) = y$. Show that this is a Verdier sequence. Show that both of these functors have adjoints

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "r"] & \mathrm{Ar}\,\mathcal{C} \arrow[l, bend right, "s"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, "t"] & \mathcal{C} \arrow[l, bend right, "q"'] \arrow[l, bend left, "\delta"]
\end{tikzcd}$$

where $\mathrm{s}(x \to y) = x$, $\mathrm{q}(y) = (0 \to y)$, and $\delta(y) = \mathrm{id}_y$. Conclude that the sequence is a split Verdier sequence.

Exercise 4.2.17. Let $f\colon R \to S$ be a morphism in $\mathrm{CAlg}(\mathrm{Sp}^{\otimes})$ and assume that f is a localisation in the sense that the multiplication map $S \otimes_R S \to S$ is an equivalence. Let $\mathrm{Mod}_R^{S-\mathrm{tors}}$ denote the full subcategory of Mod_R defined as the fiber of the extension of scalars functor $S \otimes_R -\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$. Show that

$$\mathrm{Mod}_R^{S-\mathrm{tors}} \hookrightarrow \mathrm{Mod}_R \xrightarrow{S \otimes_R -} \mathrm{Mod}_S$$

is a Verdier sequence. Exhibit adjoints of $S \otimes_R -$ and conclude that the sequence in fact is a split Verdier sequence.

Additive functors and Verdier localisations

It will be more flexible to work with squares rather than sequences in what will follow. We record the following definition:

Definition 4.2.18. Let

$$\begin{tikzcd}
\mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

be a square in $\mathrm{Cat}^{\mathrm{st}}$. We will say that

1. The square is a Verdier square if it is a pullback and both of the vertical functors are Verdier projections.
2. The square is a split Verdier square if it is a pullback and both of the vertical functors are split Verdier projections.

Observation 4.2.19. In the definition of a Verdier square above it suffices to require that the square is a pullback and that the right vertical functor is a Verdier projection since Lemma 4.2.11 ensures that the left vertical functor is also a Verdier projection. Similarly, for split Verdier squares by Lemma 4.2.15.

Observation 4.2.20. Note that a sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composite is a (split) Verdier sequence precisely if the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
0 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

is a (split) Verdier square.

Finally, we have arrived at the following central definition:

Definition 4.2.21. Let $\mathcal{A}$ be an ∞-category with finite limits. A functor

$$F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$$

with $F(0) \simeq *$ is said to be

(1) *additive* if F carries every split Verdier square to a pullback square in $\mathcal{A}$,

(2) *Verdier localising* if F carries every Verdier square to a pullback square in $\mathcal{A}$.

Notation 4.2.22. Let $\mathcal{A}$ be an ∞-category with finite limits.

1. Let $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ denote the full subcategory of $\mathrm{Fun}_*(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ spanned by those reduced functors which are additive.
2. Let $\mathrm{Fun}^{\mathrm{Ver}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ denote the full subcategory of $\mathrm{Fun}_*(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ spanned by those reduced functors which are Verdier localisations.

Note that $\mathrm{Fun}^{\mathrm{Ver}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$.

For the purpose of the following discussion it will be convenient to define additive functors using split Verdier squares rather than split Verdier sequences. Nonetheless, if we restrict our attention to additive functors which take values in stable ∞-categories, then we can ignore this distinction as the following exercise shows.

Exercise 4.2.23. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$ be a functor valued in a stable ∞-category $\mathcal{A}$ and assume that $F(0) \simeq 0$. Show that F is Verdier localising if and only if it carries Verdier sequences to fiber sequences in $\mathcal{A}$. **Hint:** *a square in a stable ∞-category is a pullback if and only if the induced map on (say vertical) fibers is an equivalence.*

We have the following structural property of additive functors:

Lemma 4.2.24. *Every additive functor $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$ preserves finite products. In particular, every such functor canonically refines to a functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{CMon}(\mathcal{A})$.*

Proof. For a pair of stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the square

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{C} \arrow[r] & 0
\end{tikzcd}$$

is split Verdier, so the map $F(\mathcal{C} \times \mathcal{D}) \to F(\mathcal{C}) \times F(\mathcal{D})$ is an equivalence. This implies the second claim since $\mathrm{CMon}(\mathcal{A}) \to \mathcal{A}$ induces an equivalence

$$\mathrm{Fun}^{\times}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{CMon}(\mathcal{A})) \to \mathrm{Fun}^{\times}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$$

by virtue of Theorem 2.3.11, since $\mathrm{Cat}^{\mathrm{st}}$ is semiadditive. ☐

Definition 4.2.25. An additive functor $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$ is grouplike if it takes values in the full subcategory $\mathrm{CGrp}(\mathcal{A}) \subseteq \mathrm{CMon}(\mathcal{A})$. Let $\mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ denote the full subcategory of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ spanned by those additive functors which are grouplike.

Example 4.2.26. The core $(-)^{\simeq} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is a Verdier localisation since it preserves arbitrary pullbacks but it is not grouplike.

We end this section with the following structural result concerning the interaction between the Q-construction and split Verdier squares.

Lemma 4.2.27. *The Q-construction functor*

$$\mathrm{Q}_n\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$$

preserves split Verdier squares for each $n \geq 0$.

Proof. Consider a split Verdier square in $\mathrm{Cat}^{\mathrm{st}}$

$$\begin{tikzcd}
\mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\
\mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

We know that p has a fully faithful left adjoint l and a fully faithful right adjoint r by Proposition 4.2.13. Using that $\mathrm{Q}_n \simeq \mathrm{Fun}(\mathcal{I}_n, -)$ for every $n \geq 0$, we obtain the adjunctions $l_* \dashv p_* \dashv r_*$, where both l_* and r_* are fully faithful by Proposition 2.1.19. By Proposition 4.2.13, we conclude that $p_*\colon \mathrm{Q}_n(\mathcal{D}) \to \mathrm{Q}_n(\mathcal{E})$ is a split Verdier projection as desired. ☐

4.3 Waldhausen's additivity theorem

In this section we establish a refinement of Waldhausen's additivity theorem. This refinement first appeared in the context of Poincaré ∞-categories (cf. [CDH+20]). We follow the presentation in [CDH+20; HW21; HLS23]. Recall that if $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is an arbitrary functor, then

$$\mathrm{Span}^F(\mathcal{C}) = \mathrm{ac}F\mathrm{Q}(\mathcal{C}) \in \mathrm{Cat}$$

We also recall that $|\mathrm{Span}^F(\mathcal{C})| \simeq |F\mathrm{Q}(\mathcal{C})|$ (cf. Remark 4.1.23).

Theorem 4.3.1 (Waldhausen additivity). *If $F : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}$ is additive, then so is*

$$|\mathrm{Span}^F(-)| : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}.$$

In particular, the functor $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}$ is additive and grouplike.

It follows from Theorem 4.3.1 that the functor $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ canonically refines to a functor $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{CGrp(An)} \simeq \mathrm{Sp}_{\geq 0}$.

Consequences

We explain how to deduce the classical variant of Waldhausen's additivity theorem.

Corollary 4.3.2. *The functor $(\mathrm{fib}, \mathrm{t}) : \mathrm{Ar}\,\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ induces an equivalence of anima*

$$\mathcal{K}(\mathrm{Ar}\,\mathcal{C}) \to \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{C})$$

for every stable ∞-category $\mathcal{C}$, where $\mathrm{t}(x \to y) = y$.

Proof. Recall Waldhausen's foundational split Verdier sequence

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "r"] & \mathrm{Ar}\,\mathcal{C} \arrow[l, bend right, "s"'] \arrow[l, bend left, "\mathrm{fib}"] \arrow[r, "t"] & \mathcal{C} \arrow[l, bend right, "q"'] \arrow[l, bend left, "\delta"]
\end{tikzcd}$$

where $\mathrm{s}(x \to y) = x$, $\mathrm{q}(y) = (0 \to y)$, and $\delta(y) = \mathrm{id}_y$. This means that for any additive and grouplike functor $F : \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}$, we obtain an equivalence

$$F(\mathrm{fib}) \times F(t)\colon F(\mathrm{Ar}\,\mathcal{C}) \overset{\simeq}{\longleftrightarrow} F(\mathcal{C}) \times F(\mathcal{C}) : F(r) + F(\delta)$$

which proves the desired statement for $F = \mathcal{K}$. ☐

Corollary 4.3.3. *Let* $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *be an additive functor and suppose that*

$$\alpha \Rightarrow \beta \Rightarrow \gamma \colon \mathcal{C} \to \mathcal{D}$$

is a cofiber sequence in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$*. Then there is an equivalence of maps of anima*

$$F\beta \simeq F\alpha + F\gamma \colon F(\mathcal{C}) \to F(\mathcal{D}).$$

Proof. Note that the functors

$$(\beta \Rightarrow \gamma), (\alpha \oplus \gamma \Rightarrow \gamma) \colon \mathcal{C} \to \mathrm{Ar}(\mathcal{D})$$

have equivalent fibers, namely α. Applying F and postcomposing with the equivalence $F(\mathrm{fib}) \times F(t) \colon F(\mathrm{Ar}(\mathcal{D}) \to \mathcal{F}(\mathcal{D}) \times F(\mathcal{D})$ yields that

$$F(\beta \Rightarrow \gamma) \simeq F(\alpha \oplus \gamma \Rightarrow \gamma) \colon F(\mathcal{C}) \to F(\mathrm{Ar}(\mathcal{D})).$$

Finally, we postcompose this equivalence with the functor $F(s) \colon F(\mathrm{Ar}(\mathcal{D})) \to F(\mathcal{D})$ which shows that $F(\beta) \simeq F(\alpha \oplus \gamma) \simeq F(\alpha) + F(\gamma)$ which proves the desired statement. □

Corollary 4.3.4 (Eilenberg swindle)**.** *Let* $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *be a grouplike and additive functor. If* $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ *admits countable direct sums, then* $F(\mathcal{C}) \simeq 0$*.*

Proof. Since $\mathcal{C}$ admits countable direct sums, we have an exact functor $\oplus_{\mathbb{N}} \colon \mathcal{C} \to \mathcal{C}$. Furthermore, the following sequence

$$\mathrm{id}_{\mathcal{C}} \Rightarrow \oplus_{\mathbb{N}} \Rightarrow \oplus_{\mathbb{N}}$$

is a cofiber sequence in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$, so we conclude that $\mathrm{id}_{F(\mathcal{C})} + F(\oplus_{\mathbb{N}}) \simeq F(\oplus_{\mathbb{N}})$ by virtue of Corollary 4.3.3. It follows that $\mathrm{id}_{F(\mathcal{C})} \simeq 0$ which proves that $F(\mathcal{C}) \simeq 0$. □

Remark 4.3.5. The Eilenberg–Swindle in Corollary 4.3.4 is the reason that we restrict to small stable ∞-categories in the definition of algebraic K-theory. Recently, the groundbreaking work of Efimov has provided an extension of algebraic K-theory to the large world of dualisable presentable stable ∞-categories.

Proof of Waldhausen's additivity theorem

We embark on the proof of Theorem 4.3.1 which relies on the following results.

Lemma 4.3.6 (Omnibus bicartesian fibration)**.** *Let* $F \in \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$.

1. *If* $p\colon \mathcal{D} \to \mathcal{E}$ *is a split Verdier projection, then* p *is a bicartesian fibration.*

2. *If* $p\colon \mathcal{C} \to \mathcal{D}$ *is an exact bicartesian fibration, then* $\mathrm{Span}^{F}(p)$ *is bicartesian.*

3. *Every bicartesian fibration is a realisation fibration, i.e., if*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

is a pullback in Cat *whose right vertical map is a bicartesian fibration, then the square remains a pullback after realisation.*

Proposition 4.3.7. *Let $F \in \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$. The functor*

$$\mathrm{Span}^F \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}$$

carries split Verdier squares to pullback squares.

We will be occupied with the proof of Lemma 4.3.6 and Proposition 4.3.7 for the remainder of this section. Before proceeding, we explain how all the pieces come together allowing us to deduce Waldhausen's additivity theorem.

Proof of Theorem 4.3.1. Let $F \colon \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}$ be an additive functor and let

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

be a split Verdier square. We have to show that $|\mathrm{Span}^F(-)|$ carries this square to a pullback. It follows from Proposition 4.3.7 that the resulting square

$$\begin{tikzcd} \mathrm{Span}^F(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{Span}^F(\mathcal{C}) \arrow[d, "\mathrm{Span}^F(p)"] \\ \mathrm{Span}^F(\mathcal{D}') \arrow[r] & \mathrm{Span}^F(\mathcal{D}) \end{tikzcd}$$

is cartesian and by Lemma 4.3.6 that the right vertical map $\mathrm{Span}^F(p)$ is a realisation fibration, which means that the square remains cartesian after realisation. This proves that $|\mathrm{Span}^F(-)|$ is additive as desired. The final claim now follows since

$$\mathcal{K} = \Omega|\mathrm{Span}(-)| = \Omega|\mathrm{Span}^{(-)^{\simeq}}(-)|$$

together with the fact that $(-)^{\simeq}$ is additive and that Ω preserves limits. □

The proof of Lemma 4.3.6 relies on the following lemma:

Lemma 4.3.8. *Let $f \colon \mathcal{A} \to \mathcal{B}$ be a functor in Cat_∞ with a left adjoint g.*

(1) *A morphism $\alpha : x \to y$ in $\mathcal{A}$ is f-cocartesian precisely if the square*

$$\begin{tikzcd}
gf(x) \arrow[r, "gf(\alpha)"] \arrow[d] & fg(y) \arrow[d] \\
x \arrow[r, "\alpha"] & y
\end{tikzcd}$$

is cocartesian, where the vertical maps are induced by the counit of the adjunction $(g \dashv f)$.

(2) *Assume that $\mathcal{A}$ admits pushouts. If f preserves pushouts and g is fully faithful, then f is a cocartesian fibration.*

Proof. For (1), note that the following square of anima

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{A}}(y, z) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{A}}(gf(y), z) \arrow[d] \\
\mathrm{Map}_{\mathcal{A}}(x, z) \arrow[r] & \mathrm{Map}_{\mathcal{A}}(gf(x), z)
\end{tikzcd}$$

is a pullback for any $z \in \mathcal{A}$ precisely if the following square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{A}}(y, z) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{B}}(f(y), f(z)) \arrow[d] \\
\mathrm{Map}_{\mathcal{A}}(x, z) \arrow[r] & \mathrm{Map}_{\mathcal{B}}(f(x), f(z))
\end{tikzcd}$$

is a pullback since g is a left adjoint of f. This precisely means that the square in (1) is a pushout if and only if α is f-cocartesian. Next, we prove (2). Let $\alpha' \colon f(x) \to y'$ be a morphism in $\mathcal{B}$. We wish to show that there exists an essentially unique f-cocartesian edge $\alpha \colon x \to y$ in $\mathcal{A}$ such that $f(\alpha) \simeq \alpha'$. Define $\alpha \colon x \to y$ by the following pushout

$$\begin{tikzcd}
gf(x) \arrow[r, "g(\alpha')"] \arrow[d, "c"] & g(y') \arrow[d] \\
x \arrow[r, "\alpha"] & y
\end{tikzcd}$$

We claim that $\alpha \colon x \to y$ is the desired f-cocartesian lift of α'. By (1), we simply have to show that $f(\alpha) \simeq \alpha'$. For this, note that gf preserves pushouts since f does by assumption and g is a left adjoint, so the square

$$\begin{tikzcd}
gfgf(x) \arrow[r, "gfg(\alpha')"] \arrow[d, "gf(c)"] & gfg(y') \arrow[d] \\
gf(x) \arrow[r, "gf(\alpha)"] & gf(y)
\end{tikzcd}$$

is a pushout. By the triangle identities and our assumption that g is fully faithful, we note that the left vertical map is an equivalence, which means that the right vertical map $gfg(y') \to gf(y)$ is an equivalence since the square is a pushout. Consequently, we find that $gf(y) \simeq gfg(y') \simeq g(y')$, where we again have used that g is fully faithful in the final equivalence. This shows that $f(y) \simeq fgf(y) \simeq fg(y') \simeq y'$ as desired. This proves that f is a cocartesian fibration. $\square$

Proof of Lemma 4.3.6. We prove (1). Let $p\colon \mathcal{D} \to \mathcal{E}$ be a split Verdier projection and apply Lemma 4.3.8 to p and its fully faitful left adjoint which exists by our assumption that p is a split Verdier projection. This shows that p is a cocartesian fibration and an entirely dual argument shows that it is also a cartesian fibration.

We prove (2). Let $p\colon \mathcal{C} \to \mathcal{D}$ be an exact bicartesian fibration. We wish to prove that $\mathrm{Span}^F(p)$ is a bicartesian fibration. This relies on the following instructive exercise:

Exercise 4.3.9. A morphism $x \leftarrow y \to z$ in $\mathrm{Span}(\mathcal{C})$ is $\mathrm{Span}(p)$-cocartesian if $y \to x$ is p-cartesian and $y \to z$ is p-cocartesian. **Hint:** *Given two solid diagrams*

$$\begin{tikzcd}
 & & w \arrow[dl, dashed] \arrow[dr, dashed] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & y \arrow[dl] \arrow[dr] & & \bullet \arrow[dl, dashed] \arrow[dr, dashed] & \\
x & & z & & v
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & p(w) \arrow[dl] \arrow[dr] & & \\
 & p(y) \arrow[dl] \arrow[dr] & & u \arrow[dl] \arrow[dr] & \\
p(x) & & p(z) & & p(v)
\end{tikzcd}$$

where the square on the right is a pullback and such that p carries the left diagram to the right diagram, we have to show that the dotted arrows and the object $\bullet$ can be filled in such a way that the square on the left is also a pullback.

Using Exercise 4.3.9 we can prove the claim for $F = (-)^{\simeq}$. Indeed, for a span $p(x) \leftarrow y' \to z'$ in $\mathcal{D}$, we can lift the left map to a p-cartesian map $y \to x$ and the right map to a p-cocartesian map $y \to z$ since p is a bicartesian fibration. Hence, the span $x \leftarrow y \to z$ in $\mathcal{C}$ is a $\mathrm{Span}(p)$-cocartesian lift of $p(x) \leftarrow y' \to z'$ by Exercise 4.3.9. This proves that $\mathrm{Span}(p)$ is a cocartesian fibration and a dual argument shows that it is also a cartesian fibration.

Next, we prove the general case when $F : \mathrm{Cat}_\infty^{\mathrm{st}} \to \mathrm{An}$ is a general additive functor instead of $(-)^{\simeq}$. Let $\mathcal{F}$ denote the full subcategory of $\mathrm{Q}_1(\mathcal{C})$ spanned by those spans where the left pointing edge is p-cartesian and the right pointing edge is p-cocartesian. The following square

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "d_1"] \arrow[d, "p"] & \mathcal{C} \arrow[d, "p"] \\
\mathrm{Q}_1(\mathcal{D}) \arrow[r, "d_1"] & \mathcal{D}
\end{tikzcd}$$

is a split Verdier square. Indeed, the square can be checked to be a pullback using Exercise 4.3.9 which is enough by our assumption that p is a split Verdier projection.

Similarly, the following square

$$\begin{tikzcd}
\mathcal{F} \times_{\mathrm{Q}_1(\mathcal{C}} \mathrm{Q}_2(\mathcal{C}) \arrow[r, "(\mathrm{id}_{\mathcal{F}}, d_1)"] \arrow[d, "p"] & \mathcal{F} \times_{\mathcal{C}} \mathrm{Q}_1(\mathcal{C}) \arrow[d, "p"] \\
\mathrm{Q}_2(\mathcal{D}) \arrow[r, "(d_2, d_1)"] & \mathrm{Q}_1(\mathcal{D}) \times_{\mathcal{D}} \mathrm{Q}_1(\mathcal{D})
\end{tikzcd}$$

where $\mathcal{F} \times_{\mathrm{Q}_1(\mathcal{C})} \mathrm{Q}_2(\mathcal{C})$ is formed using $d_2 : \mathrm{Q}_2(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C})$, is a split Verdier square. The square is once more a pullback by Exercise 4.3.9, so it remains to prove that the right vertical map is a split Verdier projection. Indeed, it is obtained as a pullback of $p : \mathrm{Q}_1(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{D})$ which is a split Verdier projection by Lemma 4.2.27, so the left vertical map is a split Verdier projection by Lemma 4.2.15.

Consequently, both of these squares are pullbacks after applying F by our assumption that F is additive. Unwinding the definitions, one can check that F applied to the second square is expressing that the essential image of

$$F(\mathcal{E}) \to F(\mathrm{Q}_1(\mathcal{C})) \to \mathrm{Ar}(\mathrm{Span}^F(\mathcal{C}))^{\simeq}$$

consists of $\mathrm{Span}^F(\mathcal{C})$-cocartesian edges, and the first diagram gives a sufficient supply of these. This proves that $\mathrm{Span}^F(p)$ is a cocartesian fibration and a dual argument proves that it is also a cartesian fibration (exchange the legs of $\mathcal{F}$). This proves (2).

Finally, we prove (3). To that end, let

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "p'"] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}' \arrow[r, "f"] & \mathcal{D}
\end{tikzcd}$$

be a pullback square in Cat, where the right vertical functor p is a bicartesian fibration. We wish to prove that the square obtained by applying $|-|$ remains a pullback square of anima. Note that p' is a cocartesian fibration since these are stable under pullback. We can rewrite the square as follows

$$\begin{tikzcd}
\mathrm{Un}(\mathrm{St}(p) \circ f) \arrow[r] \arrow[d] & \mathrm{Un}(\mathrm{St}(p)) \arrow[d] \\
\mathcal{D}' \arrow[r, "f"] & \mathcal{D}
\end{tikzcd}$$

where $\mathrm{St}(p) \colon \mathcal{D} \to \mathrm{Cat}$ is the straightening of $p \colon \mathcal{D} \to \mathcal{C}$ (cf. Construction 2.1.39). We have additionally used that $\mathrm{St}(p) \circ f \simeq \mathrm{St}(p') \colon \mathcal{D}' \to \mathrm{Cat}$.

Exercise 4.3.10. Show that every morphism in $\mathcal{D}$ is carried to a left adjoint functor by $\mathrm{St}(p) \colon \mathcal{D} \to \mathrm{Cat}$.

Using Exercise 4.3.10, if $\varphi\colon x \to y$ is a morphism in $\mathcal{D}$, then $|\mathrm{St}(p)(x)| \to |\mathrm{St}(p)(y)|$ is an equivalence, so we obtain an essentially unique factorization in the diagram below

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\mathrm{St}(p)"] \arrow[d] & \mathrm{Cat} \arrow[r, "|-|"] & \mathrm{An} \\
|\mathcal{D}| \arrow[urr, dashed, "|\mathrm{St}(p)|"'] & &
\end{tikzcd}$$

where $\mathcal{D} \to |\mathcal{D}|$ is the Dwyer–Kan localisation at all the morphisms of $\mathcal{D}$. By a similar straightening argument as above, we obtain a pullback diagram of anima

$$\begin{tikzcd}
\mathrm{Un}(|\mathrm{St}(p)| \circ |f|) \arrow[r] \arrow[d] & \mathrm{Un}(|\mathrm{St}(p)|) \arrow[d] \\
|\mathcal{D}'| \arrow[r, "|f|"] & |\mathcal{D}|
\end{tikzcd}$$

so we simply need to verify that $|\mathrm{Un}(\mathrm{St}(p) \circ f)| \simeq \mathrm{Un}(|\mathrm{St}(p)| \circ |f|)$ and $|\mathrm{Un}(\mathrm{St}(p))| \simeq \mathrm{Un}(|\mathrm{St}(p)|)$. We establish the latter and leave the former as an exercise. Using Theorem 2.1.41, we find that

$$\begin{aligned}
|\mathrm{Un}(\mathrm{St}(p))| &\simeq |\mathrm{colim}(\mathcal{D} \xrightarrow{\mathrm{St}(p)} \mathrm{Cat})| \\
&\simeq \mathrm{colim}(\mathcal{D} \xrightarrow{\mathrm{St}(p)} \mathrm{Cat} \xrightarrow{|-|} \mathrm{An}) \\
&\simeq \mathrm{colim}(|\mathcal{D}| \xrightarrow{|\mathrm{St}(p)|} \mathrm{An} \\
&\simeq \mathrm{Un}(|\mathrm{St}(p)|)
\end{aligned}$$

where we have used that realisation is a left adjoint in the second equivalence, and that $\mathcal{D} \to |\mathcal{D}|$ is colimit cofinal in the second to last equivalence. Note that we do not have to worry about inverting the cocartesian edges in Theorem 2.1.41 since every map is inverted by the realisation functor. ☐

Before proving Proposition 4.3.7, we record the following observation.

Observation 4.3.11. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. We observe that

$$F\mathrm{Q}(-)\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{sAn}$$

takes values in Segal anima. Indeed, for every $n \geq 0$ and $0 \leq i \leq n$, the square

$$\begin{tikzcd}
{[0]} \arrow[r, "0"] \arrow[d, "i"] & {[n-i]} \arrow[d, "+i"] \\
{[i]} \arrow[r] & {[n]}
\end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd}
\mathrm{Q}_n(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Q}_i(\mathcal{C}) \arrow[d] \\
\mathrm{Q}_{n-i}(\mathcal{C}) \arrow[r] & \mathrm{Q}_0(\mathcal{C})
\end{tikzcd}$$

which can be checked to be a pullback using that $\mathrm{Q}(\mathcal{C})$ is a Segal object. Furthermore, this square is split Verdier since $\mathrm{Q}_n \simeq \mathrm{Fun}(\mathcal{I}_n, -)$ so the desired fully faithful left and right adjoints of the right vertical functor are induced by left and right Kan extension of the fully faithful inclusion $\mathcal{I}_0 \hookrightarrow \mathcal{I}_n$. By our assumption that F is additive this square will be carried to a pullback square and the Segal condition follows from this by iteration.

Proof of Proposition 4.3.7. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor and let

$$\begin{tikzcd} \mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{E}' \arrow[r] & \mathcal{E} \end{tikzcd}$$

be a split Verdier square. We have to show that this square is a pullback after applying the functor $\mathrm{Span}^F(-)$. We first note that Q_n carries this square to another split Verdier sequence by Lemma 4.2.27, so we conclude that

$$\begin{tikzcd} F\mathrm{Q}(\mathcal{D}') \arrow[r] \arrow[d] & F\mathrm{Q}(\mathcal{D}) \arrow[d, "F\mathrm{Q}(p)"] \\ F\mathrm{Q}(\mathcal{E}') \arrow[r] & F\mathrm{Q}(\mathcal{E}) \end{tikzcd}$$

is a pullback of Segal anima by Observation 4.3.11 and our assumption that F is additive. We have that $\mathrm{Span}^F(p) = \mathrm{ac}F\mathrm{Q}(p)$ is a bicartesian fibration by Lemma 4.3.6, so

$$\begin{tikzcd} \mathrm{ac}F\mathrm{Q}(\mathcal{D}') \arrow[r] \arrow[d] & \mathrm{ac}F\mathrm{Q}(\mathcal{D}) \arrow[d, "\mathrm{ac}F\mathrm{Q}(p)"] \\ \mathrm{ac}F\mathrm{Q}(\mathcal{E}') \arrow[r] & \mathrm{ac}F\mathrm{Q}(\mathcal{E}) \end{tikzcd}$$

is a pullback by Lemma 4.1.20. This finishes the proof since $\mathrm{Span}^F(-) = \mathrm{ac}F\mathrm{Q}(-)$. ☐

The universal property of $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$

We have proved that the algebraic K-theory functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{st}}_{\infty} \to \mathrm{An}$$

is additive and grouplike by applying Theorem 4.3.1 to the additive functor $F = (-)^{\simeq}$. The reason for phrasing Theorem 4.3.1 with an arbitrary additive functor F in place of the core $(-)^{\simeq}$ is that it will allow us to construct a left adjoint to the inclusion of grouplike additive functors into all additive functors. This will in turn yield a universal property of $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$ which was first established in [BGT13]. We follow the presentation in [HLS23; CDH+20]. We will prove the following result:

Theorem 4.3.12. *The inclusion of* ∞*-categories*

$$\mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An}) \hookrightarrow \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$$

admits a left adjoint $(-)^{\mathrm{gp}}$ *given by the construction* $F \mapsto \Omega|\mathrm{Span}^F(-)|$.

Corollary 4.3.13. $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is the initial grouplike additive functor under* $(-)^{\simeq}$.

More generally, every additive functor $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ admits a group completion $F^{\mathrm{gp}}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ which is the initial grouplike and additive functor under F. In other words, the algebraic K-theory functor $\mathcal{K}$ is the group completion of the core $(-)^{\simeq}$.

Construction 4.3.14. Let L denote the functor of ∞-categories

$$L\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An}) \to \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$$

determined by the construction $F \mapsto \Omega|\mathrm{Span}^F(-)| \simeq \Omega|F\mathrm{Q}(-)|$. Note that L takes values in the full subcategory $\mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$ by construction.

Construction 4.3.15. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$, the inclusion $\mathcal{C} \to \mathrm{Q}_1(\mathcal{C})$ given by $x \mapsto (0 \leftarrow x \to 0)$ induces a map of anima

$$F(\mathcal{C}) \to \Omega|F\mathrm{Q}(\mathcal{C})| \simeq \Omega|\mathrm{Span}^F(\mathcal{C})|$$

as in Construction 4.1.27. Furthermore, this map is natural in $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ and thus determines a natural transformation $\eta\colon \mathrm{id} \Rightarrow L$ where both the source and target are considered as endofunctors of $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$.

The proof of Theorem 4.3.12 relies on the following result:

Proposition 4.3.16. *If* $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is grouplike and additive, then*

$$\eta_F\colon F \to LF\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is an equivalence.

Let us explain the proof of Theorem 4.3.12 using Proposition 4.3.16.

Proof of Theorem 4.3.12. Consider the functor

$$L\colon \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An}) \to \mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$$

from Construction 4.3.14. We will use Proposition 2.1.36 to prove that L is a Bousfield localisation. Recall from Construction 4.3.15 that we have the natural transformation $\eta\colon \mathrm{id} \Rightarrow L$. We note that ηL and $L\eta$ are equivalences. Indeed, the first one ηL is an equivalence by Proposition 4.3.16 and the second one is obtained as a coordinate flip of the first (cf. [HLS23, Proposition 5.3]). It follows that L is indeed a left Bousfield localisation which proves the desired statement. □

The proof of Proposition 4.3.16 relies on the décalage functor which we now recall:

Construction 4.3.17. The décalage functor $\mathrm{dec} : \mathrm{sAn} \to \mathrm{sAn}$ is induced by $[0] * - : \Delta \to \Delta$. It comes equipped with a pair of natural transformations

$$\begin{aligned} p &: \mathrm{dec} \Rightarrow \mathrm{ev}_0 \\ d_0 &: \mathrm{dec} \Rightarrow \mathrm{id} \end{aligned}$$

induced by $[0] \subseteq [0] * [n] = [1+n]$ and $[n] \subseteq [0] * [n] = [1+n]$, respectively. Note that dec is also defined on $\mathrm{sCat}^{\mathrm{st}}$.

Definition 4.3.18. For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$, define $\mathrm{Null} : \Delta^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ by

$$\mathrm{Null}(\mathcal{C}) = \mathrm{fib}_0(p : \mathrm{dec}\,\mathrm{Q}(\mathcal{C}) \to \mathrm{Q}_0(\mathcal{C}).$$

The natural transformation $d_0 : \mathrm{dec} \to \mathrm{id}$ induces a natural map $\mathrm{Null}(\mathcal{C}) \to \mathrm{Q}(\mathcal{C})$.

Example 4.3.19. An object of $\mathrm{Null}_1(\mathcal{C})$ is given by a diagram

$$\begin{tikzcd}
 & & x_{02} \arrow[dl] \arrow[dr] & & \\
 & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & \\
0 & & x_{11} & & x_{22}
\end{tikzcd}$$

in $\mathcal{C}$, where the top square is a pullback. In general, $\mathrm{Null}_n(\mathcal{C}) \subseteq \mathrm{Q}_{n+1}(\mathcal{C})$ consists of all those diagrams that vanish in the lower left corner.

Example 4.3.20. For every $n \geq 0$, let $\mathcal{I}'_n$ denote the full subposet of $\mathcal{I}_n$ spanned by those diagrams with 0 in the lower left corner. We have that $\mathrm{Null}_n(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{I}'_n, \mathcal{C})$ similarly to how $\mathrm{Q}_n(\mathcal{C}) \simeq \mathrm{Fun}(\mathcal{I}_n, \mathcal{C})$ (cf. Exercise 4.1.14).

In the course of the proof of Proposition 4.3.16 we will need to study the effect of applying a grouplike additive functor to the natural map $d_0 \colon \mathrm{Null}(\mathcal{C}) \to \mathrm{Q}(\mathcal{C})$ followed by realisation. To this end, we will use the equifibrancy lemma of Rezk.

Lemma 4.3.21 (Rezk)**.** *Let I be a small ∞-category and let*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "\tau"] \\
Z \arrow[r] & W
\end{tikzcd}$$

be a pullback in $\mathrm{Fun}(I, \mathrm{An})$ *such that τ is equifibered in the sense that*

$$\begin{tikzcd}
Y(i) \arrow[r, "\tau_i"] \arrow[d] & W(i) \arrow[d] \\
Y(j) \arrow[r, "\tau_j"] & W(j)
\end{tikzcd}$$

is a pullback for every morphism $i \to j$ in I. Then the square

$$\begin{tikzcd}
\mathrm{colim}_I X \arrow[r] \arrow[d] & \mathrm{colim}_I Y \arrow[d, "\tau"] \\
\mathrm{colim}_I Z \arrow[r] & \mathrm{colim}_I W
\end{tikzcd}$$

is a pullback as well.

Proof. See [CDH+20, Lemma 3.3.14]. □

We will apply Lemma 4.3.21 with $I = \Delta^{\mathrm{op}}$ so that $\mathrm{colim}_I \simeq |-|$.

Lemma 4.3.22. *Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be grouplike and additive. For every $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$, the natural map $d_0\colon F\mathrm{Null}(\mathcal{C}) \to F\mathrm{Q}(\mathcal{C})$ is equifibered in the sense of Lemma 4.3.21.*

Proof. By the Segal condition and that $\Delta^{\mathrm{op}}_{\mathrm{inj}} \subseteq \Delta^{\mathrm{op}}$ is colimit cofinal, we reduce to proving that the following square

$$\begin{tikzcd}
F\mathrm{Null}_2(\mathcal{C}) \arrow[r] \arrow[d, "d_i"] & F\mathrm{Q}_2(\mathcal{C}) \arrow[d, "d_i"] \\
F\mathrm{Null}_1(\mathcal{C}) \arrow[r] & F\mathrm{Q}_1(\mathcal{C})
\end{tikzcd}$$

is a pullback for $i = 0, 1, 2$. In fact, both d_1 and d_2 are split Verdier projections, so we will only treat the case when $i = 0$. Since the square above is a square of $\mathbb{E}_\infty$-groups by virtue of our assumption that F is grouplike and additive, it suffices to prove that the map between the vertical fibers over 0 is an equivalence. Before applying F, note that the counterclockwise composite $\mathrm{Null}_2(\mathcal{C}) \xrightarrow{d_0} \mathrm{Null}_1(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C})$ is given by

$$\begin{tikzcd}[column sep=tiny]
 & & & x_{03} \arrow[dl] \arrow[dr] & & & \\
 & & x_{02} \arrow[dl] \arrow[dr] & & x_{13} \arrow[dl] \arrow[dr] & & \\
 & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & & x_{23} \arrow[dl] \arrow[dr] & \\
0 & & x_{11} & & x_{22} & & x_{33}
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=tiny]
 & & x_{03} \arrow[dl] \arrow[dr] & & \\
 & x_{02} \arrow[dl] \arrow[dr] & & x_{23} \arrow[dl] \arrow[dr] & \\
0 & & x_{22} & & x_{33}
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=tiny]
 & x_{23} \arrow[dl] \arrow[dr] & \\
x_{22} & & x_{33}
\end{tikzcd}$$

while the clockwise composite $\mathrm{Null}_2(\mathcal{C}) \to \mathrm{Q}_2(\mathcal{C}) \to \mathrm{Q}_1(\mathcal{C})$ is given by

$$\begin{tikzcd}[column sep=tiny]
 & & & x_{03} \arrow[dl] \arrow[dr] & & & \\
 & & x_{02} \arrow[dl] \arrow[dr] & & x_{13} \arrow[dl] \arrow[dr] & & \\
 & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[dl] \arrow[dr] & & x_{23} \arrow[dl] \arrow[dr] & \\
0 & & x_{11} & & x_{22} & & x_{33}
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=tiny]
 & & x_{13} \arrow[dl] \arrow[dr] & & \\
 & x_{12} \arrow[dl] \arrow[dr] & & x_{23} \arrow[dl] \arrow[dr] & \\
x_{11} & & x_{22} & & x_{33}
\end{tikzcd}
\mapsto
\begin{tikzcd}[column sep=tiny]
 & x_{23} \arrow[dl] \arrow[dr] & \\
x_{22} & & x_{33}
\end{tikzcd}$$

Consequently, we see that the left vertical fiber over 0 is given by

$$\begin{tikzcd}[column sep=tiny]
 & & & 0 \arrow[dr] & & & \\
 & & 0 \arrow[ur, equal] \arrow[dl] \arrow[dr] & & x_{12} \arrow[dr] & & \\
 & x_{01} \arrow[dl] \arrow[dr] & & x_{12} \arrow[ur, equal] \arrow[dl] \arrow[dr] & & 0 \arrow[dr, equal] & \\
0 & & x_{01} \oplus x_{12} & & 0 \arrow[ur, equal] & & 0
\end{tikzcd}$$

while the right vertical fiber over 0 is given by

$$\begin{tikzcd}
 & & x_{12} \arrow[dl, equal] \arrow[dr] & & \\
 & x_{12} \arrow[dl] \arrow[dr] & & 0 \arrow[dl, equal] \arrow[dr, equal] & \\
x_{11} & & 0 & & 0
\end{tikzcd}$$

More precisely, we see that the map between the vertical fibers over 0 is induced by applying F to the functor

$$\mathcal{C} \times \mathcal{C} \to \mathrm{Ar}(\mathcal{C})$$

given by $(x, y) \mapsto (x \to x \oplus y)$. This functor is a right inverse of $(s, \mathrm{cofib})\colon \mathrm{Ar}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}$ which is an equivalence after applying F by our assumption that F is grouplike and additive. □

Finally, we prove Proposition 4.3.16 which finishes our proof of Theorem 4.3.12.

Proof of Proposition 4.3.16. First note that $|F\mathrm{Null}(\mathcal{C})| \simeq *$ for every functor F since the simplicial stable ∞-category $\mathrm{Null}(\mathcal{C})$ is split over 0. It follows that the component

$$\eta_F : F(\mathcal{C}) \to \Omega|F\mathrm{Q}(\mathcal{C})|$$

is induced by applying the functor $|F(-)|$ to the following pullback square

$$\begin{tikzcd}
\mathrm{const}\,\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Null}(\mathcal{C}) \arrow[d, "d_0"] \\
0 \arrow[r] & \mathrm{Q}(\mathcal{C})
\end{tikzcd}$$

where the top horizontal map is induced by $x \mapsto (0 \leftarrow x \to 0)$. Therefore, it suffices to prove that this square remains a pullback after applying $|F(-)|$. First note that the square above is a split Verdier square since d_0 is a split Verdier projection. Indeed, the functor $d_0\colon \mathrm{Null}(\mathcal{C}) \to \mathrm{Q}(\mathcal{C})$ has a fully faithful left and right adjoints by left and right Kan extension, respectively using that $\mathrm{Q}_n(-) \simeq \mathrm{Fun}(\mathcal{I}_n, -)$ as similarly for $\mathrm{Null}(-)$. It follows that the square of anima

$$\begin{tikzcd}
F(\mathrm{const}\,\mathcal{C}) \arrow[r] \arrow[d] & F\mathrm{Null}(\mathcal{C}) \arrow[d, "d_0"] \\
0 \arrow[r] & F\mathrm{Q}(\mathcal{C})
\end{tikzcd}$$

is a pullback since F is additive. The right vertical map is equifibered by Lemma 4.3.22, so the square is a pullback after realisation by Lemma 4.3.21. Using that $|F(\mathrm{const}\mathcal{C})| \simeq F(\mathcal{C})$, this implies that

$$\eta_F : F(\mathcal{C}) \to \Omega|F\mathrm{Q}(\mathcal{C})|$$

is an equivalence as desired using that $|F\mathrm{Null}(\mathcal{C})| \simeq *$ as explained above. □

4.4 Verdier localisation

We have established that $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive and grouplike (cf. Theorem 4.3.1). Furthermore, we proved that $\mathcal{K}$ is the initial such functor under the core $(-)^{\simeq}$ (cf. Theorem 4.3.12). The goal of this section is to show that $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ satisfies the stronger property of sending all Verdier squares to pullbacks and not just those which are split. This is the content of the following result:

Theorem 4.4.1. *The functor* $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}}_{\infty} \to \mathrm{An}$ *is Verdier localising.*

By Theorem 4.3.1, we obtain a refinement $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0} \hookrightarrow \mathrm{Sp}$. In fact, the functor $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is also a Verdier localisation which we will prove using the following exercise:

Exercise 4.4.2. Prove that a fiber sequence $x \to y \to z$ in $\mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$ is a fiber sequence of spectra if and only if $\pi_0(y) \to \pi_0(z)$ is surjective.

We now have the following consequence of Theorem 4.4.1:

Corollary 4.4.3. *The functor* $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ *is Verdier localising.*

Proof. Since Sp is stable, it suffices to show that $\mathcal{K}$ carries Verdier sequences in $\mathrm{Cat}^{\mathrm{st}}$ to fiber sequence in Sp by Exercise 4.2.23. If $\mathcal{C} \hookrightarrow \mathcal{D} \to \mathcal{E}$ is a Verdier sequence, then

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D}) \to \mathcal{K}(\mathcal{E})$$

is a fiber sequence in CGrp(An) by Theorem 4.4.1, so it suffices to show that $\pi_0\mathcal{K}(\mathcal{D}) \to \pi_0\mathcal{K}(\mathcal{E})$ is surjective by Exercise 4.4.2. Indeed, this follows from the explicit formula for $\pi_0\mathcal{K}$ together with the fact that Verdier projections are essentially surjective (cf. Corollary 4.2.10). ☐

Remark 4.4.4. Note that Theorem 4.4.1 is still not quite enough to establish Quillen's localisation sequences. Recall that

$$\mathrm{Mod}_{\mathbb{Z}}^{p-\mathrm{tors}} \hookrightarrow \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[1/p]}$$

is a Verdier sequence since $\mathbb{Z} \to \mathbb{Z}[1/p]$ is a localisation (cf. Exercise 4.2.17). It follows that $\mathcal{K}$ carries this sequence to a fiber sequence but the resulting fiber sequence is the zero fiber sequence by virtue of the Eilenberg swindle (cf. Corollary 4.3.4). We will later see how to salvage this using the so-called Karoubi sequences instead.

In the setting of Theorem 4.4.1, our goal is to prove that if $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a Verdier sequence, then $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D}) \to \mathcal{K}(\mathcal{E})$ is a fiber sequence. First, we will establish the following result which provides a general formula for the cofiber of $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D})$ whenever $\mathcal{C} \hookrightarrow \mathcal{D}$ is a stable subcategory. The proof of Theorem 4.4.1 will then proceed by comparing the desired cofiber $\mathcal{K}(\mathcal{E})$ with the formula afforded by the result. Let us first introduce a bit of notation.

Notation 4.4.5. For a stable subcategory $\mathcal{C} \subseteq \mathcal{D}$, let $\mathrm{Fun}^{\mathcal{C}}(I, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(I, \mathcal{D})$ spanned by those functors $I \to \mathcal{D}$ which carry each map in I to an equivalence modulo $\mathcal{C}$ (a map in $\mathcal{D}$ whose (co)fiber lies in $\mathcal{C}$).

The following result is (in some form) due to Waldhausen and is often referred to as Waldhausen's generic fibration theorem.

Theorem 4.4.6. *Let $i \colon \mathcal{C} \subseteq \mathcal{D}$ be a stable subcategory and let $G \in \mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$. The functors* const: $\mathcal{D} \to \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D})$ *induce a bifiber sequence of $\mathbb{E}_\infty$-groups*

$$G(\mathcal{C}) \to G(\mathcal{D}) \to |G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}))|,$$

where the right-most term denotes the realisation of the simplicial $\mathbb{E}_\infty$-group given by the construction $[n] \mapsto G(\mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}))$.

Sifted colimits in $\mathrm{CGrp}(\mathrm{An})$ are preserved by the forgetful functor $\mathrm{CGrp}(\mathrm{An}) \to \mathrm{An}$, so the right hand term in Theorem 4.4.6 is simply the realisation of the simplicial anima $G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}))$. We invite the reader to do the following instructive exercise which we will need later on:

Exercise 4.4.7. Let $M \to N \to K$ be a sequence in $\mathrm{CGrp}(\mathrm{An})$ with a chosen nullhomotopy of the composite. Prove that $M \to N \to K$ is a bifiber sequence in $\mathrm{CGrp}(\mathrm{An})$ if and only if the underlying sequence of anima is a fiber sequence over the unit of K and the map $\pi_0 N \to \pi_0 K$ is surjective.

With these remarks out of the way, we prove Theorem 4.4.6 following [HLS23].

Proof. We indicate a proof using the relative Q-construction following [HLS23, §8]. We define the relative Q-construction $\mathrm{Q}(i) \colon \Delta^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$ by the following pullback

$$\begin{tikzcd}
\mathrm{Q}(i) \arrow[r] \arrow[d] & \mathrm{Null}(\mathcal{D}) \arrow[d] \\
\mathrm{Q}(\mathcal{C}) \arrow[r] & \mathrm{Q}(\mathcal{D})
\end{tikzcd}$$

First note that the right vertical functor is a split Verdier projection as in the proof of Proposition 4.3.16. Recall that $G\mathrm{Null}(\mathcal{D}) \to G\mathrm{Q}(\mathcal{D})$ is equifibered (cf. Lemma 4.3.22), so the square

$$\begin{tikzcd}
|G\mathrm{Q}(i)| \arrow[r] \arrow[d] & |G\mathrm{Null}(\mathcal{D})| \arrow[d] \\
|G\mathrm{Q}(\mathcal{C})| \arrow[r] & |G\mathrm{Q}(\mathcal{D})|
\end{tikzcd}$$

is cartesian, which means that

$$|G\mathrm{Q}(i)| \to |G\mathrm{Q}(\mathcal{C})| \to |G\mathrm{Q}(\mathcal{D})|$$

is a fiber sequence since $|G\mathrm{Null}(\mathcal{D})| \simeq *$. Rotating this fiber sequence twice to the left and using Proposition 4.3.16 gives the desired fiber sequence of $\mathbb{E}_\infty$-groups

$$G(\mathcal{C}) \to G(\mathcal{D}) \to |G\mathrm{Q}(i)|$$

since one can check that $|G\mathrm{Q}(i)| \simeq |G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}))|$ (cf. [HLS23, Proposition 8.3]). Finally, to check that it is also a cofiber sequence it suffices to check that $\pi_0|G\mathrm{Q}(\mathcal{C})| = 0$ by the long exact sequence. But the functor $\mathcal{C} \to \mathrm{Q}_1(\mathcal{C})$ defined by $x \mapsto (0 \leftarrow 0 \to x)$ induces a homotopy between the 0 and the identity maps of $\pi_0|G\mathrm{Q}(\mathcal{C})|$. □

Note that the projection $\mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}) \to \mathrm{Fun}([n], \mathcal{D}/\mathcal{C})$ takes values in the subcategory spanned by those functors taking every morphism of $[n]$ to equivalences in $\mathcal{D}/\mathcal{C}$ but this subcategory is equivalent to the essential image of the fully faithful functor $\mathrm{const}\colon \mathcal{D}/\mathcal{C} \to \mathrm{Fun}([n], \mathcal{D}/\mathcal{C})$ which provides a map of simplicial ∞-categories

$$\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}) \to \mathrm{const}\, \mathcal{D}/\mathcal{C},$$

which in turn induces a map of $\mathbb{E}_\infty$-groups

$$|G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}))| \to G(\mathcal{D}/\mathcal{C})$$

for every grouplike and additive functor $G\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$. This allows us to use Theorem 4.4.6 to prove the following bootstrap result:

Proposition 4.4.8. *Let $G \in \mathrm{Fun}^{\mathrm{gp}}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})$. The following are equivalent:*

1. *The functor $G\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is a Verdier localisation which satisfies that*

$$\pi_0 G(\mathcal{D}) \to \pi_0 G(\mathcal{E})$$

is a surjection for every Verdier projection $\mathcal{D} \to \mathcal{E}$.

2. *The canonical map of $\mathbb{E}_\infty$-groups*

$$|G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to G(\mathcal{D}/\mathcal{C})$$

is an equivalence for every Verdier inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$.

Proof. We first prove that (1) $\Rightarrow$ (2). Assume that $G\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is a Verdier localisation which carries Verdier projections to π_0-surjections. Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a Verdier inclusion. Since G is Verdier localising, we obtain a fiber sequence of $\mathbb{E}_\infty$-groups

$$G(\mathcal{C}) \to G(\mathcal{D}) \to G(\mathcal{D}/\mathcal{C})$$

which is also a fiber sequence of spectra since $\pi_0 G(\mathcal{D}) \to \pi_0 G(\mathcal{D}/\mathcal{C})$ is surjective by assumption (cf. Exercise 4.4.2). Similarly, we obtain a bifiber sequence of $\mathbb{E}_\infty$-groups

$$G(\mathcal{C}) \to G(\mathcal{D}) \to |G(\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})|$$

by virtue of Theorem 4.4.6 which is also a fiber sequence of spectra since $\pi_0 G(\mathcal{D}) \to \pi_0|G(\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})|$ can be checked to be surjective. We see that the following diagram of fiber sequences of spectra

$$\begin{tikzcd}
G(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{id}"] & G(\mathcal{D}) \arrow[r] \arrow[d, "\mathrm{id}"] & {|G(\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})|} \arrow[d] \\
G(\mathcal{C}) \arrow[r] & G(\mathcal{D}) \arrow[r] & G(\mathcal{D}/\mathcal{C})
\end{tikzcd}$$

yields that the right vertical map of spectra is a π_*-isomorphism, where we again have used that $\pi_0 G(\mathcal{D}) \to \pi_0|G(\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})|$ is surjective to treat the case of π_0. This establishes (2). We prove that (2) ⇒ (1). Assume that $|G(\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})| \to G(\mathcal{D}/\mathcal{C})$ is an equivalence. It follows from Theorem 4.4.6 that

$$G(\mathcal{C}) \to G(\mathcal{D}) \to G(\mathcal{D}/\mathcal{C})$$

is a bifiber sequence in CGrp(An), which implies that the underlying sequence of anima is a fiber sequence over the unit of $G(\mathcal{D}/\mathcal{C})$ and the map $\pi_0 G(\mathcal{D}) \to \pi_0 G(\mathcal{D}/\mathcal{C})$ is surjective by Exercise 4.4.7. To summarize, we have shown that G carries Verdier sequences to fiber sequences of anima. To show that G is a Verdier localisation, we need to show that G carries Verdier squares to pullback squares. A square in CGrp(An) is a pullback square if and only if the maps on all vertical fibers are equivalences. Since G carries Verdier projections to π_0-surjections, we see that all fibers are non-empty and are equivalent to the fiber over 0. Finally, we note that the induced map on fibers over 0 is an equivalence since F carries Verdier sequences to fiber sequences as argued above. □

Now Proposition 4.4.8 allows us to prove the following bootstrap result:

Proposition 4.4.9. *Let* $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *be an additive functor satisfying the following:*

(∗) *The canonical map of anima*

$$|F(\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})| \to F(\mathcal{D}/\mathcal{C})$$

is an equivalence for every Verdier inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$.

Then the functor

$$F^{\mathrm{gp}} = \Omega|\mathrm{Span}^F(-)| \simeq \Omega|F\mathrm{Q}(-)|\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is a Verdier localisation which carries Verdier projections to π_0*-surjection.*

Proof. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ denote an additive functor and assume that F satisfies (∗). Let $G = F^{\mathrm{gp}} = \Omega|F\mathrm{Q}(-)|\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ and recall that G is additive and grouplike by Theorem 4.3.1. We wish to show that G satisfies that the canonical map of anima

$$|G\,\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})| \to G(\mathcal{D}/\mathcal{C})$$

is an equivalence for every Verdier inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ since this would prove the desired claim by virtue of Proposition 4.4.8. In other words, we wish to prove that

$$|F\mathrm{Q}\,\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})| \to |F\mathrm{Q}(\mathcal{D}/\mathcal{C})|$$

is an equivalence. Using the identification

$$\mathrm{Q}\,\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{Q}(\mathcal{C})}([-],\mathrm{Q}(\mathcal{D})),$$

it suffices to prove that the relevant map is an equivalence in each simplicial degree. Hence, we wish to prove that the canonical map of anima

$$|F\,\mathrm{Fun}^{\mathrm{Q}_k(\mathcal{C})}([-],\mathrm{Q}_k(\mathcal{D}))| \to |F\mathrm{Q}_k(\mathcal{D}/\mathcal{C})|$$

is an equivalence for each k. For every k, the sequence

$$\mathrm{Q}_k(\mathcal{C}) \to \mathrm{Q}_k(\mathcal{D}) \to \mathrm{Q}_k(\mathcal{D}/\mathcal{C})$$

is a Verdier sequence using that $\mathrm{Q}_k \simeq \mathrm{Fun}(\mathcal{I}_k,-)$ and that $\mathcal{I}_k$ is a finite poset. Consequently, we may reduce to the case when $k=0$, where the desired map is an equivalence by virtue of our assumption that F satisfies $(*)$. This proves the desired statement. □

To deduce Theorem 4.4.1 we wish to apply Proposition 4.4.9 to the additive functor given by the core $F=(-)^{\simeq}$. To that end, if $\mathcal{C} \hookrightarrow \mathcal{D}$ is a Verdier inclusion, then

$$\mathrm{Fun}^{\mathcal{C}}([-],\mathcal{D})^{\simeq} \simeq \mathrm{Fun}([-],\mathcal{D}_{\mathcal{C}})^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}}([-],\mathcal{D}_{\mathcal{C}}) \simeq \mathrm{N}(\mathcal{D}_{\mathcal{C}}),$$

where $\mathcal{D}_{\mathcal{C}}$ denotes the ∞-category of equivalences in $\mathcal{D}$ modulo $\mathcal{C}$ (cf. Definition 4.2.3), and N is the Rezk nerve from Construction 4.1.17. Note that we only used that $\mathcal{C}$ is a stable subcategory for this. Note that $\mathcal{D}_{\mathcal{C}}$ consists of those maps in $\mathcal{D}$ which are sent to equivalences by the Verdier projection $\mathcal{D} \to \mathcal{D}/\mathcal{C}$, so we obtain a map $\mathcal{D}_{\mathcal{C}} \to (\mathcal{D}/\mathcal{C})^{\simeq}$ which in turn induces a map $|\mathcal{D}_{\mathcal{C}}| \to (\mathcal{D}/\mathcal{C})^{\simeq}$ by the universal property of Dwyer–Kan localisations. Finally, we note that $|\mathrm{N}(\mathcal{D}_{\mathcal{C}})| \to |\mathcal{D}_{\mathcal{C}}|$ is an equivalence, since

$$|\mathrm{N}(\mathcal{D}_{\mathcal{C}})| \simeq |\mathrm{acN}(\mathcal{D}_{\mathcal{C}})| \xrightarrow{\simeq} |\mathcal{D}_{\mathcal{C}}|,$$

where we have used in the last step that the Rezk nerve is fully faithful, so the claim follows from:

Proposition 4.4.10. *The canonical map*

$$|\mathcal{D}_{\mathcal{C}}| \to (\mathcal{D}/\mathcal{C})^{\simeq}$$

is an equivalence for every Verdier inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$*. In other words, the functor* $(-)^{\simeq}$ *satisfies* $(*)$*.*

Proof. The map in question is induced by the following localisation map

$$|\mathcal{D}_{\mathcal{C}}| = \mathcal{D}_{\mathcal{C}}[(\mathcal{D}_{\mathcal{C}})^{-1}] \to \mathcal{D}[(\mathcal{D}_{\mathcal{C}})^{-1}] = \mathcal{D}/\mathcal{C}.$$

Note that the datum of such a map is equivalent to the datum of a map $|\mathcal{D}_{\mathcal{C}}| \to (\mathcal{D}/\mathcal{C})^{\simeq}$ by adjunction. In [HLS23, Proposition 6.8], it is proved by explicit calculations with calculus of fractions a la Gabriel–Zisman in the homotopy categories of the localisations that this map is an equivalence. The crucial input is that $\mathcal{D}_{\mathcal{C}}$ actually satisfies the 2-out-of-6 axiom rather than the 2-out-of-3 axiom. □

The proof of Theorem 4.4.1 is now an immediate consequence of the results above.

Proof of Theorem 4.4.1. Proposition 4.4.10 ensures that $(-)^{\simeq}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ satisfies the condition of Proposition 4.4.9, so we conclude that

$$\mathcal{K} \simeq (-)^{\simeq,\mathrm{gp}} \simeq \Omega|\mathrm{Q}(-)^{\simeq}|\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is a Verdier localisation. □

Remark 4.4.11. In the setting of Proposition 4.4.8, we may also conclude that if F satisfies $(*)$, then F^{gp} is not just a Verdier localisation but it also carries Verdier projections to π_0-surjections. From this, we may also conclude that $F^{\mathrm{gp}}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is a Verdier localisation by the same argument as in Corollary 4.4.3. However, in the proof of Corollary 4.4.3, we checked that $\mathcal{K}$ carries Verdier projections to π_0-surjections by hand but we might as well have deduced this from the discussion above.

4.5 Karoubi localisation

In the previous section, we proved that the functor

$$\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is a Verdier localisation (cf. Theorem 4.4.1). Recall that we have a Verdier sequence

$$\mathrm{Mod}_{\mathbb{Z}}^{p-\mathrm{tors}} \hookrightarrow \mathrm{Mod}_{\mathbb{Z}} \xrightarrow{\mathbb{Z}[1/p]\otimes_{\mathbb{Z}}-} \mathrm{Mod}_{\mathbb{Z}[1/p]},$$

so $\mathcal{K}$ carries this sequence to a fiber sequence of anima. However, this fiber sequence is the zero sequence by virtue of the Eilenberg swindle. Instead, to obtain our desired fiber sequence we would have to consider the following sequence

$$\mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{tors}} \hookrightarrow \mathrm{Perf}_{\mathbb{Z}} \xrightarrow{\mathbb{Z}[1/p]\otimes_{\mathbb{Z}}-} \mathrm{Perf}_{\mathbb{Z}[1/p]},$$

where we have restricted to perfect modules. Unfortunately, this is no longer a Verdier sequence. There are two different reasons for why this might fail. In our case, we have

$$\mathrm{Perf}_{\mathbb{Z}} / \mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{tors}} \to \mathrm{Perf}_{\mathbb{Z}[1/p]}$$

is not an equivalence. However, we will see that the map above is an equivalence after idempotent completion. The second reason for why this might fail is the following: If $R \to S$ is a localisation of $\mathbb{E}_\infty$-rings, then the base-change functor

$$S \otimes_R - \colon \operatorname{Perf}_R \to \operatorname{Perf}_S$$

might fail to be essentially surjective which in particular excludes the possibility that $S \otimes_R -$ is a Verdier projection. This is more subtle and we mention that $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} -$ is essentially surjective. We will discuss this further in Example 4.5.13.

Karoubi sequences

The notion of a Karoubi sequence is a variant of Verdier sequences which additionally are invariant under addition of direct summands or idempotent completion. We have the following central definition:

Definition 4.5.1. An exact functor of stable ∞-categories $\mathcal{C} \to \mathcal{D}$ is a *Karoubi equivalence* if it is fully faithful and has dense image, in the sense that every object of $\mathcal{D}$ is a retract of an object of the essential image.

Amazingly, Thomason gave a complete classification of the possible Karoubi equivalences to a fixed stable category.

Theorem 4.5.2 (Thomason's Galois correspondence, [Tho97, Thm. 2.1], [CDH+20, Thm. A.3.2])**.** *Let $\mathcal{D}$ be a small stable category. Then there is a bijection between the subgroups of $\mathcal{D}$ and the Karoubi equivalences to $\mathcal{D}$: this is given by assigning a Karoubi equivalence $f \colon \mathcal{C} \to \mathcal{D}$ to the subgroup* $\operatorname{Im}(f) \leq \mathrm{K}_0(\mathcal{D})$.

Example 4.5.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ and recall that the idempotent completion $\mathcal{C}^\natural$ is the retract closure of $\mathcal{C}$. Furthermore, we have that $\mathcal{C}$ is idempotent complete precisely if every idempotent in $\mathcal{C}$ splits (cf. [Lur12, §4.4.5]). Moreover, every $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$ admits a smallest dense stable subcategory given by

$$\mathcal{C}^{\min} = \{x \in \mathcal{C} \mid 0 = [x] \in \pi_0\mathcal{K}(\mathcal{C})\}.$$

This is a consequence of Thomason's Galois correspondence between stable dense subcategories of $\mathcal{C}$ and subgroups of $\pi_0\mathcal{K}(\mathcal{C})$ (cf. [Tho97]). The assignment $\mathcal{C} \mapsto \mathcal{C}^{\min}$ is functorial. The minimalisation functor will not play an essential role for us.

The following exercise provides an instructive way of becoming acquainted with the definitions.

Exercise 4.5.4. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. Prove the following assertions:

(*i*) Karoubi equivalences are closed under 2-out-of-3.

(*ii*) The functor $\mathcal{C} \hookrightarrow \mathcal{C}^\natural$ is a Karoubi equivalence.

(*iii*) The functor $\mathcal{C}^{\min} \hookrightarrow \mathcal{C}$ is a Karoubi equivalence. **Hint:** *Show that every $x \in \mathcal{C}$ is a retract of $x \oplus \Sigma x$ and argue that the latter is an element of $\mathcal{C}^{\min}$.*

Let $\mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{st}}$. Prove that the following are equivalent:

(*i*) The functor $\mathcal{C} \to \mathcal{D}$ is a Karoubi equivalence.

(*ii*) The induced functor $\mathcal{C}^\natural \to \mathcal{D}^\natural$ on idempotent completions is an equivalence.

(*ii*) The induced functor $\mathcal{C}^{\min} \to \mathcal{D}^{\min}$ on minimalisations is an equivalence.

Proposition 4.5.5. *The Dwyer–Kan localisation of $\mathrm{Cat}^{\mathrm{st}}$ at the Karoubi equivalences is a left and right Bousfield localisation. The right adjoint is determined by $\mathcal{C} \mapsto \mathcal{C}^\natural$, and the left adjoint is determined by $\mathcal{C} \mapsto \mathcal{C}^{\min}$.*

Proof. See [CDH+20, Proposition A.3.3]. □

We will use the following notation:

Notation 4.5.6. Let $\mathrm{Cat}^{\mathrm{perf}}$ denote the full subcategory of $\mathrm{Cat}^{\mathrm{st}}$ spanned by those small stable ∞-categories which are idempotent complete.

Proposition 4.5.5 provides a pair of adjunctions

$$\begin{tikzcd}
\mathrm{Cat}^{\mathrm{st}} \arrow[r] & \mathrm{Cat}^{\mathrm{st}}[\mathrm{Kar}^{-1}] \arrow[l, bend right=50, hook', "(-)^{\min}"'] \arrow[l, bend left=50, hook, "(-)^{\natural}"]
\end{tikzcd}$$

where Kar denotes the collection of Karoubi equivalences and both of the adjoints $(-)^\natural$ and $(-)^{\min}$ are fully faithful. It follows that the idempotent completion functor furnishes an equivalence $(-)^\natural \colon \mathrm{Cat}^{\mathrm{st}}[\mathrm{Kar}^{-1}] \xrightarrow{\simeq} \mathrm{Cat}^{\mathrm{perf}}$. Similarly, we note that $(-)^{\min}$ determines an equivalence between $\mathrm{Cat}^{\mathrm{st}}[\mathrm{Kar}^{-1}]$ and the full subcategory of $\mathrm{Cat}^{\mathrm{st}}$ spanned by those small stable ∞-categories $\mathcal{C}$ with $\pi_0\mathcal{K}(\mathcal{C}) \simeq 0$.

Finally, we define the notion of a Karoubi sequence:

Definition 4.5.7. A sequence of stable ∞-categories and exact functors

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$$

with vanishing composite is a *Karoubi sequence* if the sequence

$$\mathcal{C}^\natural \xrightarrow{i^\natural} \mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural$$

it is both a fiber and a cofiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$. We say that i is a *Karoubi inclusion* and that p is a *Karoubi projection.*

Remark 4.5.8. Equivalently, we might ask the sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ to be both a fiber and a cofiber sequence in the Dwyer–Kan localisation $\mathrm{Cat}^{\mathrm{st}}[\mathrm{Kar}^{-1}]$.

As in our discussion of Verdier sequence, we have the following concrete characterisation of Karoubi sequences.

Proposition 4.5.9. *Let* $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *be a sequence in* $\mathrm{Cat}^{\mathrm{st}}$ *with vanishing composite. The following are equivalent:*

(*i*) *The sequence* $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *is a Karoubi sequence.*

(*ii*) *The functor* i *is fully faithful and the functor* $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ *is a Karoubi equivalence.*

Proof. We prove $(i) \Rightarrow (ii)$. Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Karoubi sequence which means that

$$\mathcal{C}^\natural \xrightarrow{i^\natural} \mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural$$

is both a fiber and cofiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$ by definition. The functor $(-)^\natural \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}$ preserves limits and colimits and $\mathrm{Cat}^{\mathrm{perf}}$ is closed under limits in $\mathrm{Cat}^{\mathrm{st}}$, thus

$$\mathrm{fib}(\mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural) \simeq \mathrm{fib}(\mathcal{D} \xrightarrow{p} \mathcal{E})^\natural.$$

This proves that $\mathcal{C}^\natural \xrightarrow{i^\natural} \mathcal{D}^\natural \xrightarrow{p^\natural} \mathcal{E}^\natural$ being a fiber sequence is equivalent to $\mathcal{C} \to \mathrm{fib}(\mathcal{D} \to \mathcal{E})$ being a Karoubi equivalence. Similarly, we observe that the sequence above being a cofiber sequence is equivalent to the functor $\mathcal{D}/i(\mathcal{C}) \to \mathcal{E}$ being a Karoubi equivalence. This proves $(i) \Rightarrow (ii)$. Conversely, assume that $i \colon \mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful and that $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is a Karoubi equivalence. We have that

$$\mathrm{fib}(\mathcal{D} \to \mathcal{E}) \simeq \mathrm{fib}(\mathcal{D} \to \mathcal{D}/\mathcal{C})$$

which by Lemma 4.2.7 means that $i \colon \mathcal{C} \to \mathrm{fib}(\mathcal{D} \to \mathcal{D}/\mathcal{C})$ has dense essential image and therefore is a Karoubi equivalence. This proves that $(ii) \Rightarrow (i)$. ☐

Example 4.5.10. Every Verdier sequence is a Karoubi sequence.

We obtain the following immediate consequence of Proposition 4.5.9.

Corollary 4.5.11. *Let* $i \colon \mathcal{C} \to \mathcal{D}$ *and* $p \colon \mathcal{D} \to \mathcal{E}$ *be functors in* $\mathrm{Cat}^{\mathrm{st}}$.

(*i*) *The functor* i *is a Karoubi inclusion if and only if it is fully faithful.*

(*ii*) *The functor* p *is a Karoubi projection if and only if the functor* $p \colon \mathcal{D} \to p(\mathcal{D})$ *is a Verdier projection and* $p(\mathcal{D})$ *is dense.*

Remark 4.5.12. It follows from Corollary 4.5.11 that a Karoubi projection is a Verdier projection if and only if it is essentially surjective.

Example 4.5.13. Let $f \colon R \to S$ be a localisation of $\mathbb{E}_\infty$-rings. For the following discussion to make sense we will need to assume that f has perfectly generated fiber which is a property we will return to later on. In this case, the base-change functor

$$S \otimes_R - \colon \mathrm{Perf}_R \to \mathrm{Perf}_S$$

is essentially surjective precisely if $\mathcal{K}_0(R) \to \mathcal{K}_0(S)$ is surjective. The latter is for instance the case for $\mathbb{Z} \to \mathbb{Z}[1/p]$. We refer the reader to [CDH+20, Example A.4.6] for a counterexample.

Lemma 4.5.14. *Karoubi projections are stable under pullback.*

Proof. See [CDH+20, Lemma A.3.10]. □

Finally, we define Karoubi squares and Karoubi localisations in parallel to how we previously defined Verdier squares and Verdier localisations.

Definition 4.5.15. A square in $\mathrm{Cat}^{\mathrm{st}}$

$$\begin{tikzcd}
\mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

is a Karoubi square if it is a pullback in $\mathrm{Cat}^{\mathrm{st}}[\mathrm{Kar}^{-1}]$ and the vertical functors are Karoubi projections. Equivalently, that the square is a pullback in $\mathrm{Cat}^{\mathrm{perf}}$ after idempotent completion $(-)^{\natural}$.

Observation 4.5.16. In the definition of a Karoubi square above it suffices to require that the right vertical functor is a Karoubi projection since Lemma 4.5.14 ensures that the left vertical functor is also a Karoubi projection.

Observation 4.5.17. Note that a sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composite is a Karoubi sequence precisely if the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
0 \arrow[r] & \mathcal{E}
\end{tikzcd}$$

is a Karoubi square.

Finally, we have arrived at the following central definition:

Definition 4.5.18. Let $\mathcal{A}$ be an ∞-category with finite limits. A reduced functor

$$F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$$

is *Karoubi localising* if F carries every Karoubi square to a pullback square in $\mathcal{A}$.

Notation 4.5.19. Let $\mathcal{A}$ be an ∞-category with finite limits. Let $\mathrm{Fun}^{\mathrm{Kar}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ denote the full subcategory of $\mathrm{Fun}_*(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$ spanned by those reduced functors which are Karoubi localisations. Note that $\mathrm{Fun}^{\mathrm{Kar}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A}) \subseteq \mathrm{Fun}^{\mathrm{Ver}}(\mathrm{Cat}^{\mathrm{st}}, \mathcal{A})$.

We end with the following useful exercise:

Exercise 4.5.20. Let $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{A}$ be a reduced functor. Prove that F is a Karoubi localisation if and only if it is a Verdier localisation and inverts Karoubi equivalences.

The cofinality theorem and K-theory as a Karoubi localisation

We wish to prove the following theorem following [HLS23].

Theorem 4.5.21. *The functor* $\mathcal{K} \circ (-)^\natural \colon \mathrm{Cat}^{\mathrm{st}}_\infty \to \mathrm{An}$ *is Karoubi localising.*

To prove this result, we will first establish a bootstrap result which allows us to construct Karoubi localisations from Verdier localisations. Namely, we have the following:

Lemma 4.5.22. *Let* $G \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *be a Verdier localisation. If* G *carries pullbacks in* $\mathrm{Cat}^{\mathrm{st}}$*, whose vertical legs are dense inclusions, to pullbacks of anima, then the functor*

$$G \circ (-)^\natural \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is a Karoubi localisation.

Proof. Recall from Exercise 4.5.20 that a reduced functor is a Karoubi localisation if and only if it is a Verdier localisation and carries Karoubi equivalences to equivalences. Since $(-)^\natural$ carries Karoubi equivalences to equivalences, it suffices to prove that

$$G \circ (-)^\natural \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is again Verdier localising. To that end, let

$$\begin{tikzcd}
\mathcal{D}' \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

be a Verdier square and let F denote the common vertical fiber. Since the idempotent completion functor $(-)^\natural$ preserves limits by Proposition 4.5.5, we conclude that

$$\begin{tikzcd}
(\mathcal{D}')^\natural \arrow[r] \arrow[d] & \mathcal{D}^\natural \arrow[d] \\
(\mathcal{E}')^\natural \arrow[r] & \mathcal{E}^\natural
\end{tikzcd}$$

is a pullback, so we have to verify that G carries this square to a pullback of anima. Using the common vertical fiber F, we may rewrite the diagram as follows:

$$\begin{tikzcd}
(\mathcal{D}')^\natural \arrow[r] \arrow[d] & \mathcal{D}^\natural \arrow[d] \\
(\mathcal{D}')^\natural/F \arrow[r] \arrow[d] & \mathcal{D}^\natural/F \arrow[d] \\
(\mathcal{E}')^\natural \arrow[r] & \mathcal{E}^\natural
\end{tikzcd}$$

We show that G carries both the top and the bottom square to a pullback of anima. The top square is a Verdier square, so it is carried to a pullback by G since G is Verdier localising. Secondly, we claim that the vertical functors in the bottom square are dense inclusions. Indeed, this follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}/F \arrow[r] \arrow[d, "\simeq"] & \mathcal{D}^\natural/F \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}^\natural
\end{tikzcd}$$

by the 2-out-of-3 property for dense inclusions since equivalences are dense inclusions and idempotent completions are dense inclusions and therefore also Verdier quotients of such. By our assumption on G, it therefore remains to check that the bottom square is also a pullback. Using the formula for mapping anima in pullbacks and that the vertical maps are fully faithful, we see that the canonical functor

$$(\mathcal{D}')^\natural/F \to (\mathcal{E}')^\natural \times_{\mathcal{E}^\natural} \mathcal{D}^\natural/F$$

is fully faithful. We also see that this functor is essentially surjective using that $\mathcal{D}^\natural \to \mathcal{D}^\natural/F$ is essentially surjective and that the total square is a pullback. This proves the desired statement. □

Since $\mathcal{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is a Verdier localisation (cf. Theorem 4.4.1), we need to verify that $\mathcal{K}$ satisfies the assumption of Lemma 4.5.22 above. This will be a consequence of the so-called cofinality theorem:

Theorem 4.5.23 (Cofinality)**.** *If $\mathcal{C} \hookrightarrow \mathcal{D}$ is a dense inclusion, then*

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

is an isomorphism for $i \geq 1$ and there is a short exact sequence

$$0 \to \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}) \to \pi_0(\mathcal{D}^\simeq)/\pi_0(\mathcal{C}^\simeq) \to 0.$$

Proof. The π_0-statement is due to Thomason [Tho97], so we prove that $\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$ is an isomorphism for $i \geq 1$. By Theorem 4.4.6, we have a fiber sequence

$$|\mathrm{Q}(\mathcal{C})^\simeq| \to |\mathrm{Q}(\mathcal{D})^\simeq| \to ||\operatorname{Fun}^{\mathrm{Q}(\mathcal{C})}([-], \mathrm{Q}(\mathcal{D}))^\simeq||,$$

where we again have used the identification of bisimplicial anima

$$\mathrm{Q}\operatorname{Fun}^{\mathcal{C}}([-], \mathcal{D})^\simeq \simeq \operatorname{Fun}^{\mathrm{Q}(\mathcal{C})}([-], \mathrm{Q}(\mathcal{D}))^\simeq.$$

As in the discussion before Proposition 4.4.10, we conclude that

$$|\operatorname{Fun}^{\mathrm{Q}_k(\mathcal{C})}([-], \mathrm{Q}_n(\mathcal{D}))^\simeq| \simeq |\mathrm{Q}_n(\mathcal{D})_{\mathrm{Q}_n(\mathcal{C})}|$$

for every n. The point is now that since $\mathcal{C} \to \mathcal{D}$ is dense, so is $\mathrm{Q}_n(\mathcal{C}) \to \mathrm{Q}_n(\mathcal{D})$ which in turn implies that $\mathrm{Q}_n(\mathcal{D})/\mathrm{Q}_n(\mathcal{C}) \simeq 0$. Using Proposition 4.4.10, we conclude that $|\mathrm{Q}_n(\mathcal{D})_{\mathrm{Q}_n(\mathcal{C})}|$ is discrete with

$$\pi_0|\mathrm{Q}_n(\mathcal{D})_{\mathrm{Q}_n(\mathcal{C})}| \simeq \pi_0(\mathrm{Q}_n(\mathcal{D})^{\simeq})/\pi_0(\mathrm{Q}_n(\mathcal{C})^{\simeq}) \simeq \mathcal{K}_0(\mathrm{Q}_n(\mathcal{D}))/\mathcal{K}_0(\mathrm{Q}_n(\mathcal{C}))$$

where we have used Thomason's result in the final isomorphism. For $\mathcal{E} \in \mathrm{Cat}^{\mathrm{st}}$ one can now explicitly check that $\mathcal{K}_0(\mathrm{Q}(\mathcal{E}))$ is the edgewise subdivision of $\mathrm{Bar}(\mathcal{K}_0(\mathcal{E}))$, so

$$||\,\mathrm{Fun}^{\mathrm{Q}(\mathcal{C})}([-], \mathrm{Q}(\mathcal{D}))^{\simeq}|| \simeq |\mathrm{Bar}(\mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C})|.$$

The latter is an Eilenberg–Mac Lane space in degree 1, so looping the fiber sequence from the beginning proves the desired statement. $\square$

Using the cofinality theorem in combination with the bootstrap result above, we can now finally prove Theorem 4.5.21. We need the following instructive exercise.

Exercise 4.5.24. Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be dense. Show that the following are satisfied:

(i) The map $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is an inclusion of path components.

(ii) For every $x \in \pi_0(\mathcal{D}^{\simeq})$ there is $x' \in \pi_0(\mathcal{D}^{\simeq})$ such that $x + x' \in \pi_0(\mathcal{C}^{\simeq})$.

(iii) An $x \in \pi_0(\mathcal{D}^{\simeq})$ lies in $\pi_0(\mathcal{C}^{\simeq})$ if there exists $y \in \pi_0(\mathcal{C}^{\simeq})$ such that $x+y \in \pi_0(\mathcal{C}^{\simeq})$.

Deduce that there is a short exact sequence of commutative monoids

$$0 \to \pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})/\pi_0(\mathcal{C}^{\simeq}) \to 0.$$

Proof of Theorem 4.5.21. By Lemma 4.5.22 we to prove that $\mathcal{K}$ preserves pullbacks

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense inclusions. Note that the square

$$\begin{tikzcd} \pi_0(\mathcal{C}'^{,\simeq}) \arrow[r] \arrow[d] & \pi_0(\mathcal{C}^{\simeq}) \arrow[d] \\ \pi_0(\mathcal{D}'^{,\simeq}) \arrow[r] & \pi_0(\mathcal{D}^{\simeq}) \end{tikzcd}$$

is a pullback since $(-)^{\simeq}$ preserves pullbacks and the left vertical map is injective by virtue of Exercise 4.5.24. We show that this implies that also $\mathcal{K}_0(-)$ preserves the pullback above. This will then prove the desired statement by virtue of Theorem 4.5.23.

Since $\mathcal{C}' \to \mathcal{D}'$ is dense, we obtain a short exact sequence of commutative monoids

$$0 \to \pi_0(\mathcal{C}'^{,\simeq}) \to \pi_0(\mathcal{D}'^{,\simeq}) \to \pi_0(\mathcal{D}'^{,\simeq})/\pi_0(\mathcal{C}'^{,\simeq}) \to 0$$

by Exercise 4.5.24 from which a diagram-chase reveals that the induced map

$$\pi_0(\mathcal{D}'^{,\simeq})/\pi_0(\mathcal{C}'^{,\simeq}) \to \pi_0(\mathcal{D}^{\simeq})/\pi_0(\mathcal{C}^{\simeq})$$

is injective. It then follows from Theorem 4.5.23 that the induced map

$$\mathcal{K}_0(\mathcal{D}')/\mathcal{K}_0(\mathcal{C}') \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C})$$

is also an injection, which proves that $\mathcal{K}_0(-)$ preserves the original pullback as desired.

□

The Thomason–Neeman localisation theorem and applications

In this section, we will finally produce examples of Karoubi sequences. In particular, we will establish that Quillen's celebrated localisation sequence

$$\mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{tors}} \to \mathrm{Perf}_{\mathbb{Z}} \to \mathrm{Perf}_{\mathbb{Z}[1/p]}$$

is a Karoubi sequence, yielding the fiber sequence $\mathcal{K}(\mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{tors}}) \to \mathcal{K}(\mathbb{Z}) \to \mathcal{K}(\mathbb{Z}[1/p])$. To do so, we will need to make a short digression on inductive completions which is a procedure for freely adjoining filtered colimits to an ∞-category. We refer the reader to [Lur12, §5.3.5] for a more comprehensive treatment.

Definition 4.5.25. Let $\mathcal{C}$ be an ∞-category with finite colimits and let $\mathrm{Ind}(\mathcal{C})$ denote the full subcategory of $\mathrm{PSh}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ spanned by those functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ which preserve finite limits.

Notation 4.5.26. The Yoneda embedding $y\colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ factors over $\mathrm{Ind}(\mathcal{C})$ and we let $y_\omega\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ denote the restricted Yoneda embedding.

Remark 4.5.27. Note that $\mathrm{Ind}(\mathcal{C})$ is closed under filtered colimits in $\mathrm{PSh}(\mathcal{C})$ since these are calculated levelwise and filtered colimits commute with finite limits in anima.

Remark 4.5.28. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. There is an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}).$$

induced by postcomposition with $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$. Consequently, we find that

$$\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}).$$

whenever $\mathcal{C}$ is stable. In this case it also follows that $\mathrm{Ind}(\mathcal{C})$ is again stable.

The ∞-category $\mathrm{Ind}(\mathcal{C})$ is characterized by the following universal property:

Proposition 4.5.29. *Let $\mathcal{D}$ denote an ∞-category which admits filtered colimits. The restricted Yoneda embedding y_ω induces an equivalence*

$$\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}^{\omega}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve filtered colimits.

Proof. See [Lur12, Proposition 5.3.5.10]. □

For every functor $f\colon \mathcal{C} \to \mathcal{D}$, let $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$ denote the functor obtained as the left Kan extension of f along the restricted Yoneda $y_\omega\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d, "y_\omega"] & \mathcal{D} \\
\mathrm{Ind}(\mathcal{C}) \arrow[ur, dashed, "f_!"'] &
\end{tikzcd}$$

We use the universal property of $\mathrm{Ind}(-)$ to obtain the following extra functoriality:

Construction 4.5.30. For every functor $f\colon \mathcal{C} \to \mathcal{D}$ which preserves finite colimits, we obtain a functor

$$f^*\colon \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C})$$

by precomposition with f using that $\mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$. This functor admits a left adjoint $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ which is induced by the composite

$$\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{y_\omega} \mathrm{Ind}(\mathcal{D}).$$

by Proposition 4.5.29. We denote this left adjoint by $\mathrm{Ind}(f)\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$. If $\mathcal{C}$ and $\mathcal{D}$ are assumed to be stable, then all the functors discussed above are exact.

Lemma 4.5.31. *Suppose that $\mathcal{D}$ admits finite colimits. The following are equivalent:*

(*i*) *There is a small ∞-category $\mathcal{C}$ such that $\mathcal{D} \simeq \mathrm{Ind}(\mathcal{C})$.*

(*ii*) *The ∞-category $\mathcal{D}$ admits filtered colimits and has a set of compact generators.*

If $\mathcal{D}$ satisfies the equivalent conditions above, then $\mathcal{C} \simeq \mathcal{D}^\omega$, where $\mathcal{D}^\omega$ denotes the full subcategory of compact objects of $\mathcal{D}$.

Proof. See [HW21, Lemma IV.33c]. □

Exercise 4.5.32. Let R denote an $\mathbb{E}_\infty$-ring. Show that $\mathrm{Mod}_R \simeq \mathrm{Ind}(\mathrm{Perf}_R)$ by proving that $\{R[i] \mid i \in \mathbb{Z}\}$ is a set of compact generators of Mod_R.

Example 4.5.33. We have that $\mathrm{Ind}(\mathcal{C})^\omega \simeq \mathcal{C}^\natural$ by virtue of [Lur12, Lemma 5.4.2.4]. In other words, the restricted Yoneda embedding $y_\omega\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})^\omega$ exhibits the target as the idempotent completion of $\mathcal{C}$.

Example 4.5.34. The ∞-category Mod_R is idempotent complete since it admits small colimits (arbitrary direct sums suffices). Since Perf_R is the full subcategory of compact R-modules it follows that Perf_R is also idempotent complete.

Finally, we arrive at localisation theorem of Thomason–Neeman [Nee92]:

Theorem 4.5.35. *A sequence $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composite is a Karoubi sequence precisely if the induced sequence*

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{\mathrm{Ind}(i)} \mathrm{Ind}(\mathcal{D}) \xrightarrow{\mathrm{Ind}(p)} \mathrm{Ind}(\mathcal{E})$$

is a Verdier sequence.

Remark 4.5.36. We have only defined Verdier sequences for small stable ∞-categories and inductive completions are in general not small. Using Proposition 4.2.9, we can still make sense of Verdier sequences in the setting of large ∞-categories, namely:

1. The functor $\mathrm{Ind}(i)$ is fully faithful and the essential image is closed under retracts.
2. The functor $\mathrm{Ind}(p)$ exhibits $\mathrm{Ind}(\mathcal{E})$ as the Verdier quotient of $\mathrm{Ind}(\mathcal{D})$ by $\mathrm{Ind}(\mathcal{C})$.

The proof of Theorem 4.5.35 relies on the following result which says that we can detect Karoubi equivalences on Ind-categories.

Lemma 4.5.37. *Let $i\colon \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}^{\mathrm{st}}$. The following are equivalent:*

- *(i) The functor $i\colon \mathcal{C} \to \mathcal{D}$ is a Karoubi equivalence.*
- *(ii) The left Kan extension $i_!\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is an equivalence.*

Proof. We first prove that $(i) \Rightarrow (ii)$. Let $i\colon \mathcal{C} \to \mathcal{D}$ be a Karoubi equivalence and recall that we wish to prove that the left Kan extension

$$i_!\colon \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$$

is an equivalence, where we have used Remark 4.5.28 since we are working in the stable situation. Since $\mathcal{C} \hookrightarrow \mathcal{D}$ is fully faithful, we conclude that the left Kan extension $i_!$ is fully faithful, so checking that $i_!$ is an equivalence amounts to proving that

$$i^*\colon \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$$

is an equivalence. Equivalences can be checked on $(-)^{\simeq}$ and $\mathrm{Ar}(-)^{\simeq}$. To verify that i^* is an equivalence on $(-)^{\simeq}$, note that we have a diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Cat}^{\mathrm{perf}}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "i^*"] \arrow[d, "\simeq"] & \mathrm{Map}_{\mathrm{Cat}^{\mathrm{perf}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathrm{Cat}^{\mathrm{st}}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}) \arrow[r, "i^*"] & \mathrm{Map}_{\mathrm{Cat}^{\mathrm{st}}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})
\end{tikzcd}$$

where the vertical maps are given by the equivalences

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{perf}}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}) \simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{st}}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp}^{\natural}) \simeq \mathrm{Map}_{\mathrm{Cat}^{\mathrm{st}}}(\mathcal{D}^{\mathrm{op}}, \mathrm{Sp})$$

afforded by Proposition 4.5.5 and the fact that Sp is idempotent complete. Since $\mathcal{C} \to \mathcal{D}$ is a Karoubi equivalence, we conclude that the top horizontal map is an equivalence which implies that the lower horizontal map is an equivalence. This is precisely the assertion that i^* is an equivalence on $(-)^{\simeq}$. The argument for $\mathrm{Ar}(-)^{\simeq}$ is similar.

We prove that *(ii)* $\Rightarrow$ *(i)*. Assume that $i\colon \mathcal{C} \to \mathcal{D}$ is functor in $\mathrm{Cat}^{\mathrm{st}}$ and that the left Kan extension $i_!\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is an equivalence. We wish to prove that i is a Karoubi equivalence. From the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[d, hook, "\omega"] & \mathcal{D} \arrow[d, hook, "y_\omega"] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "\simeq"] & \mathrm{Ind}(\mathcal{D})
\end{tikzcd}$$

we observe that $i\colon \mathcal{C} \to \mathcal{D}$ is fully faithful. Next, we wish to prove that every element of $\mathcal{D}$ can be written as a retract of an object of $\mathcal{C}$. Let $d \in \mathcal{D}$ and note that d canonically refines to an object of $\mathrm{Ind}(\mathcal{C})$ by the diagram above. Hence, we obtain an equivalence

$$d \xrightarrow{\simeq} \mathrm{colim}_{i \in \mathcal{I}} c_i,$$

where $\mathcal{I}$ is filtered and $c_i \in \mathcal{C}$ for every $i \in \mathcal{I}$. Since d is a compact object of $\mathrm{Ind}(\mathcal{D})$, we conclude that the equivalence $d \to \mathrm{colim}_{i \in \mathcal{I}} c_i$ factors over c_{i_0} for some $i_0 \in \mathcal{I}$, so

$$d \to c_{i_0} \to \mathrm{colim}_{i \in \mathcal{I}} c_i \xrightarrow{\simeq} d$$

exhibits d as a retract of $c_{i_0} \in \mathcal{C}$ as desired. This finishes the proof. □

We will also use the following result which says that Ind preserves being fully faithful. In general, left Kan extensions of fully faithful functors along fully faithful functors need not be fully faithful.

Lemma 4.5.38. *If $f\colon \mathcal{C} \to \mathcal{D}$ is fully faithful, then the left Kan extension*

$$f_!\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$$

is fully faithful.

Proof. By our assumption that f is fully faithful, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "f"] \arrow[d, hook, "y_\omega"] & \mathcal{D} \arrow[d, hook, "y_\omega"] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "f_!"] & \mathrm{Ind}(\mathcal{D})
\end{tikzcd}$$

For $x, y \in \mathrm{Ind}(\mathcal{C})$ we may write $x \simeq \mathrm{colim}_{i \in I} x_i$ and $y \simeq \mathrm{colim}_{j \in J} y_j$, where both I and J are filtered and $x_i, y_j \in \mathcal{C}$. We wish to prove that the induced map

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(x, y) \to \mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(f_! x, f_! y)$$

is an equivalence. This map is equivalently given by

$$\lim_i \mathrm{colim}_j \, \mathrm{Map}_{\mathcal{C}}(x_i, y_j) \to \lim_i \mathrm{colim}_j \, \mathrm{Map}_{\mathcal{D}}(f(x_i), f(y_j))$$

which in turn is induced by the map

$$\mathrm{Map}_{\mathcal{C}}(x_i, y_j) \to \mathrm{Map}_{\mathcal{D}}(f(x_i), f(y_j))$$

which is an equivalence since f is fully faithful. This proves the desired statement. □

Proof of Theorem 4.5.35. Since we will only use one direction of the result, we refer the reader to [HW21, Theorem IV.34] for a proof of the direction that Karoubi sequences induce Verdier sequences on Ind-categories. Let $\mathcal{C} \xrightarrow{i} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be in $\mathrm{Cat}^{\mathrm{st}}$ so that

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{i_!} \mathrm{Ind}(\mathcal{D}) \xrightarrow{p_!} \mathrm{Ind}(\mathcal{E})$$

is a Verdier sequence. Let $\mathcal{C}' = \mathrm{fib}(\mathcal{D} \to \mathcal{E})$ and $\mathcal{E}' = \mathcal{D}/\mathcal{C}'$. We claim that both $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{E}' \to \mathcal{E}$ are Karoubi equivalences. Note that $\mathcal{C}' \to \mathcal{D} \to \mathcal{E}'$ is a Verdier sequence by construction, so the induced sequence $\mathrm{Ind}(\mathcal{C}') \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E}')$ is also a Verdier sequence. By considering the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[rr, hook] \arrow[dr] & & \mathrm{Ind}(\mathcal{D}) \\
& \mathrm{Ind}(\mathcal{C}') \arrow[ur, hook] &
\end{tikzcd}$$

we conclude that $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ is fully faithful since $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ is by our assumption and $\mathrm{Ind}(\mathcal{C}') \to \mathrm{Ind}(\mathcal{D})$ is by Lemma 4.5.38. The functor $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}')$ is also essentially surjective since $\mathrm{Ind}(\mathcal{C})$ is the fiber of $\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$, so $\mathcal{C} \to \mathcal{C}'$ is a Karoubi equivalence by Lemma 4.5.37.

We prove that $\mathcal{E}' \to \mathcal{E}$ is a Karoubi equivalence. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}') \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ind}(\mathcal{D}) \arrow[r] \arrow[d, equal] & \mathrm{Ind}(\mathcal{E}') \arrow[d] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r] & \mathrm{Ind}(\mathcal{D}) \arrow[r] & \mathrm{Ind}(\mathcal{E})
\end{tikzcd}$$

where the lower sequence is also Verdier since $\mathcal{C}' \to \mathcal{D} \to \mathcal{E}'$ is a Verdier sequence. It follows that the right vertical map $\mathrm{Ind}(\mathcal{E}) \to \mathrm{Ind}(\mathcal{E}')$ is an equivalence since the left vertical map is an equivalence as $\mathcal{C} \to \mathcal{C}'$ is a Karoubi equivalence as argued above. This proves that $\mathcal{E} \to \mathcal{E}'$ is a Karoubi equivalence by Lemma 4.5.37.

We conclude that $\mathcal{C} \to \mathcal{D}$ is fully faithful and that $\mathcal{D}/\mathcal{C} \to \mathcal{E}$ is a Karoubi equivalence by considering the following diagram

$$\begin{tikzcd}
\mathcal{D}/\mathcal{C} \arrow[r] \arrow[d, "\simeq"] & \mathcal{E} \\
\mathcal{D}/\mathcal{C}' \arrow[r, equal] & \mathcal{E}' \arrow[u]
\end{tikzcd}$$

which exhibits the top horizontal map as a composite of Karoubi equivalences. This proves that $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a Karoubi sequence by Proposition 4.5.9. □

Finally we use Theorem 4.4.1 to establish Quillen's celebrated localisation sequence.

Exercise 4.5.39. Prove that the following are equivalent:

1. A map of $\mathbb{E}_\infty$-rings $f\colon R \to S$ is a localisaiton ($S \otimes_R S \to S$ is an equivalence).
2. The R-module given by $I = \mathrm{fib}(f\colon R \to S)$ satisfies that $S \otimes_R I \simeq 0$.

hIn other words, $f\colon R \to S$ is a localisation precisely $I \in \mathrm{Mod}_R^{S-\mathrm{tors}}$.

Definition 4.5.40. A $f\colon R \to S$ localisation of $\mathbb{E}_\infty$-rings has perfectly generated fiber if I lies in the smallest subcategory $\mathrm{Mod}_R^{S-\mathrm{tors}}$ containing $\mathrm{Perf}_R^{S-\mathrm{tors}}$ and which is closed under colimits.

Exercise 4.5.41. Let R denote an $\mathbb{E}_\infty$-ring and let $x \in \pi_0(R)$. Show that

$$\mathrm{fib}(R \to R[x^{-1}]) \simeq \mathrm{colim}_{n \in \mathbb{N}} R/x^n,$$

where R/x^n denotes the R-module defined by $R/x^n = \mathrm{cofib}(x^n\colon R \to R)$. Deduce that $R \to R[x^{-1}]$ has perfectly generated fiber. Similarly, prove that $R \to R[S^{-1}]$, where S is a multiplicative subset of $\pi_0(R)$, has perfectly generated fiber.

We have the following result:

Proposition 4.5.42. *If $R \to S$ is a localisation with perfectly generated fiber, then*

$$\mathrm{Perf}_R^{S-\mathrm{tors}} \to \mathrm{Perf}_R \xrightarrow{S \otimes_R -} \mathrm{Perf}_S$$

is a Karoubi sequence.

Proof. By virtue of Theorem 4.5.35 and Exercise 4.5.32 it suffices to prove that

$$\mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}}) \to \mathrm{Mod}_R \xrightarrow{S \otimes_R -} \mathrm{Mod}_S$$

is a Verdier sequence, so in turn it suffices to prove that

$$\mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}}) \simeq \mathrm{Mod}_R^{S-\mathrm{tors}}$$

under the assumption that the fiber I of $R \to S$ is perfectly generated since

$$\mathrm{Mod}_R^{S-\mathrm{tors}} \hookrightarrow \mathrm{Mod}_R \xrightarrow{S \otimes_R -} \mathrm{Mod}_S$$

is a Verdier sequence by Exercise 4.2.17.

The fully faithful functor $\mathrm{Perf}_R^{S-\mathrm{tors}} \hookrightarrow \mathrm{Perf}_R$ induces a fully faithful functor

$$\mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}}) \hookrightarrow \mathrm{Ind}(\mathrm{Perf}_R) \simeq \mathrm{Mod}_R$$

whose essential image is contained in $\mathrm{Mod}_R^{S-\mathrm{tors}}$ since the full subcategory of S-torsion R-modules is closed under colimits. Furthermore, the essential image is closed under colimits since it is closed under filtered colimits by construction of Ind and by finite colimits since it is a stable subcategory. We have proved that $\mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}})$ contains $\mathrm{Perf}_R^{S-\mathrm{tors}}$ and closed under colimits, so it follows that $I \in \mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}})$ since the fiber of $R \to S$ is perfectly generated. Consequently, we may write I as a colimit

$$I = \mathrm{colim}_{i\in\mathcal{I}} P_i,$$

where $\mathcal{I}$ is filtered and $P_i \in \mathrm{Perf}_R^{S-\mathrm{tors}}$ for every $i \in \mathcal{I}$. Let us begin the proof proper. Let $M \in \mathrm{Mod}_R^{S-\mathrm{tors}}$. We wish to prove that $M \in \mathrm{Ind}(\mathrm{Perf}_R^{S-\mathrm{tors}})$. We may write

$$M = \mathrm{colim}_{j\in\mathcal{J}} Q_j,$$

where $\mathcal{J}$ is filtered and $Q_j \in \mathrm{Perf}_R$ for every $j \in \mathcal{J}$. Since M is S-torsion, we conclude that the R-module homomorphism $M \otimes_R I \to M$ is an equivalence since it cofiber is $M \otimes_R S \simeq 0$. Therefore, we conclude that

$$\begin{aligned} M &\simeq I \otimes_R M \\ &\simeq \mathrm{colim}_{i\in\mathcal{I}} P_i \otimes_R \mathrm{colim}_{j\in\mathcal{J}} Q_j \\ &\simeq \mathrm{colim}_{(i,j)\in\mathcal{I}\times\mathcal{J}} P_i \otimes_R Q_j. \end{aligned}$$

since $I \otimes_R -$ preserves colimits. To finish the proof, we argue that $P_i \otimes_R Q_j \in \mathrm{Perf}_R^{S-\mathrm{tors}}$ since this would M as a filtered colimit of S-torsion perfect R-modules. First note that $P_i \otimes_R Q_j$ is S-torsion since $S \otimes_R -$ preserves colimits. It remains to show that $P_i \otimes_R Q_j$ is compact as an R-module which holds since

$$\begin{aligned} \mathrm{Map}_R(P_i \otimes_R Q_j, \mathrm{colim}_k M_k) &\simeq P_i^\vee \otimes_R Q_j^\vee \otimes_R \mathrm{colim}_k M_k \\ &\simeq \mathrm{colim}_k (P_i^\vee \otimes_R Q_j^\vee \otimes_R M_k) \\ &\simeq \mathrm{colim}_k \, \mathrm{Map}_R(R, P_i^\vee \otimes_R Q_j^\vee \otimes_R M_k) \\ &\simeq \mathrm{colim}_k \, \mathrm{Map}_R(P_i \otimes_R Q_j, M_k) \end{aligned}$$

for every filtered colimit, where we have used that both P_i and Q_i are dualisable (cf. Exercise 4.1.3). This proves the desired statement. □

Corollary 4.5.43. *If $R \to S$ is a localisation of $\mathbb{E}_\infty$-rings with perfectly generated fiber, then there is a fiber sequence of anima*

$$\mathcal{K}(\mathrm{Perf}_R^{S-\mathrm{tors}}) \to \mathcal{K}(R) \to \mathcal{K}(S)$$

which is a fiber sequence of spectra precisely if $\mathcal{K}_0(R) \to \mathcal{K}_0(S)$ is surjective.

Proof. We have that $\mathrm{Perf}_R^{S-\mathrm{tors}} \to \mathrm{Perf}_R \to \mathrm{Perf}_S$ is a Karoubi sequence by Proposition 4.5.42, so $\mathcal{K}$ carries the sequence above to the desired fiber sequence of anima by virtue of Theorem 4.5.21. The final assertion follows from Exercise 4.4.2. □

Example 4.5.44. Recall that $\mathbb{Z} \to \mathbb{Z}[1/p]$ is a localisation with perfectly generated fiber by Exercise 4.5.41. Consequently, we obtain our desired fiber sequence of spectra

$$\mathcal{K}(\mathrm{Perf}_{\mathbb{Z}}^{p-\mathrm{tors}}) \to \mathcal{K}(\mathbb{Z}) \to \mathcal{K}(\mathbb{Z}[1/p])$$

since $\mathbb{Z} \to \mathbb{Z}[1/p]$ is a $\mathcal{K}_0$-surjection. We similarly obtain the fiber sequence of spectra

$$\mathcal{K}(\mathrm{Perf}_{\mathbb{Z}}^{\mathrm{tors}}) \to \mathcal{K}(\mathbb{Z}) \to \mathcal{K}(\mathbb{Q}).$$

Remark 4.5.45. The assumption that $R \to S$ has perfectly generated fiber ensures that $\mathrm{Mod}_R^{S-\mathrm{tors}}$ is a compactly generated stable ∞-category which means that $\mathcal{K}$ is a well-defined operation. Recently, Efimov has introduced an extension of $\mathcal{K}$ to the dualisable setting which agrees with the usual $\mathcal{K}$ for compactly generated categories. Using this one can remove the assumption that the fiber of $R \to S$ is perfectly generated.

Toward nonconnective K-theory

In Theorem 4.5.21, we proved that the functor

$$\mathcal{K} \circ (-)^{\natural} \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is a Karoubi localisation. This gives rise to nonconnective algebraic K-theory by means of the following result:

Theorem 4.5.46. *The functor*

$$\Omega^{\infty} \colon \mathrm{Fun}(\mathrm{Cat}_{\infty}^{\mathrm{st}}, \mathrm{Sp}) \to \mathrm{Fun}(\mathrm{Cat}_{\infty}^{\mathrm{st}}, \mathrm{An})$$

restricts to an equivalence $\mathrm{Fun}^{\mathrm{Kar}}(\mathrm{Cat}_{\infty}^{\mathrm{st}}, \mathrm{Sp}) \simeq \mathrm{Fun}^{\mathrm{Kar}}(\mathrm{Cat}_{\infty}^{\mathrm{st}}, \mathrm{An})$.

In other words, there is a Karoubi localisation

$$\mathrm{K} \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$$

which satisfies that $\Omega^{\infty}\mathrm{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{C}^{\natural})$. This is called nonconnective algebraic K-theory. We will not prove Theorem 4.5.46 but we will indicate a construction of nonconnective K-theory. Before doing so, let us discuss the following example which essentially tells us that K satisfies Zariski descent on commutative rings.

Example 4.5.47. Let R be a commutative ring and assume that $\mathrm{Spec}(R) = U \cup V$ is an open Zariski cover. This means that $U = \mathrm{Spec}(R[x^{-1}])$ and $V = \mathrm{Spec}(R[y^{-1}])$ for elements $x, y \in R$ such that $(x) + (y) = (1)$. We claim that the square

$$\begin{tikzcd}
\mathrm{K}(R) \arrow[r] \arrow[d] & \mathrm{K}(R[x^{-1}]) \arrow[d] \\
\mathrm{K}(R[y^{-1}]) \arrow[r] & \mathrm{K}(R[x^{-1}, y^{-1}])
\end{tikzcd}$$

is a pullback of spectra. Indeed, the map on horizontal fibers is given by

$$\mathrm{K}(\mathrm{Perf}_R^{x-\mathrm{tors}}) \to \mathrm{K}(\mathrm{Perf}_{R[y^{-1}]}^{x-\mathrm{tors}})$$

which is an equivalence already before taking K-theory by virtue of the relation $(x)+(y)=(1)$ which shows that if x acts nilpotently, then y acts invertibly. We remark that an entirely similar argument shows that K satisfies Nisnevich descent on commutative rings.

We end with a construction of nonconnective algebraic K-theory.

Example 4.5.48. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and note that the following sequence

$$\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})^{\omega} \to \mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C}$$

is a Karoubi sequence. Consequently, we obtain a fiber sequence of connective spectra

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathrm{Ind}(\mathcal{C})^{\omega}) \to \mathcal{K}(\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C})$$

but $\mathcal{K}(\mathrm{Ind}(\mathcal{C})^{\omega}) \simeq 0$ since $\mathrm{Ind}(\mathcal{C})^{\omega}$ has countable colimits. It follows that

$$\mathcal{K}(\mathcal{C}) \simeq \Omega\mathcal{K}(\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C}).$$

Note that $\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C}$ might fail to be idempotent complete, so we cannot iterate this procedure to obtain further deloopings of $\mathcal{K}(\mathcal{C})$. However, Theorem 4.5.23 ensures that the induced map

$$\mathcal{K}(\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C}) \to \mathcal{K}((\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C})^{\natural})$$

is a 1-connective cover. Define $\mathrm{Calk}(\mathcal{C}) = (\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C})^{\natural}$, so we get that

$$\mathcal{K}(\mathcal{C}) \simeq \tau_{\geq 0}\Omega\mathcal{K}(\mathrm{Calk}(\mathcal{C}))$$

for an idempotent complete stable ∞-category $\mathcal{C}$. Crucially, we note that $\mathcal{K}_0(\mathrm{Calk}(\mathcal{C}))$ might be non-zero and we in fact have that

$$\mathrm{K}_{-1}(\mathcal{C}) = \mathcal{K}_0(\mathrm{Calk}(\mathcal{C})).$$

We will turn this into a definition as follows:

Definition 4.5.49. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and define the Calkin ∞-category by

$$\mathrm{Calk}(\mathcal{C}) = (\mathrm{Ind}(\mathcal{C})^{\omega}/\mathcal{C})^{\natural}.$$

Inductively define $\mathrm{Calk}^0(\mathcal{C}) = \mathcal{C}^{\natural}$ and $\mathrm{Calk}^{n+1}(\mathcal{C}) = \mathrm{Calk}(\mathrm{Calk}^n(\mathcal{C}))$.

Exercise 4.5.50. Show that there is an equivalence of connective spectra

$$\tau_{\geq 0}\Omega\mathcal{K}(\mathrm{Calk}^{n+1}(\mathcal{C})) \simeq \mathcal{K}(\mathrm{Calk}^n(\mathcal{C}))$$

for every $n \geq 0$.

Definition 4.5.51. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$. The nonconnective K-theory spectrum $\mathrm{K}(\mathcal{C})$ is the spectrum with $\tau_{\geq -n}\mathrm{K}(\mathcal{C}) = \Omega^n\mathcal{K}(\mathrm{Calk}^n(\mathcal{C}))$. Equivalently, $\mathrm{K}(\mathcal{C}) \simeq \mathrm{colim}\tau_{\geq -n}\mathrm{K}(\mathcal{C})$.

Observation 4.5.52. The canonical map $\mathcal{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ is a connective cover precisely if $\mathcal{C}$ is idempotent complete.

5 Algebraic K–theory: the family tree and devissage

So far, we have seen in great detail in Chapter 4 the various kinds of descent properties that K–theory satisfies. While this elucidates how one can "glue" old computations to get new ones, we still fall far short in terms of being able to produce any primitive, irreducible computation to input into the gluing machine.

In this chapter, we will introduce perhaps the most general version of the algebraic K–theory functor which takes as input the so–called "Waldhausen categories" (c.f. §5.2 for the definition, as well as for the special case of *exact* categories). Even if one is only interested in K–theory of stable categories, this level of generality is still crucial to the general theory since it provides the setting in which to speak of the K–theory of additive and abelian categories for instance, where "calculations from first principles" are more possible by virtue of Grothendieck and Quillen's powerful method of *devissage*. We will prove many classical results here which allow one to relate the K–theories of stable categories to the K–theories of these more computationally amenable categories and indicate various incarnations of these devissage arguments. We follow the comprehensive set of lecture notes [Win24] by Christoph Winges for many parts of our presentation.

5.1 A family tree

A family tree, for readers who might not have seen such things, is a diagram consisting of members of a family and how they are related to each other. In this sense, we will list in this section several key results in the literature that relate the K–theories of various distinct Waldhausen categories. This chapter concerns the proofs of some of them, and we will point the reader to [Win24] for those results that we will not cover in this lecture series.

The first result we mention, which will be proved in §5.3, concerns the bridge between the K–theory studied in Chapter 4 and the group–completion K–theory from Chapter 3. The result is as follows:

Theorem 5.1.1 (Waldhausen's +=S, Theorem 5.3.17)**.** *Let $\mathcal{D}$ be split exact category. Then there is a canonical equivalence* $\mathrm{k}(\mathcal{D}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{D}) \in \mathrm{CGrp}$.

Examples of split exact categories include the category Proj_R of projective R–modules for classical rings R. This is a very powerful result since we have seen several methods to compute $\mathrm{k}(R)$ using classical homological computations, and indeed Theorem 5.1.1 will serve as one third of the ingredient that will allow us to compute $\mathrm{K}(R)$ in the presence of the localisation theorems from Chapter 4.

The remaining ingredients will be supplied to us by Grothendieck and Quillen's famous method of *devissage*, the basics of which we will see in §5.4. The second ingredient will then come in the form of the following:

Theorem 5.1.2 (Quillen's Devissage, Theorem 5.4.1)**.** *Let $i\colon \mathcal{A} \subseteq \mathcal{B}$ be an exact inclusion of abelian categories such that $\mathcal{A}$ is closed under subobjects in $\mathcal{B}$. Suppose furthermore that any $B \in \mathcal{B}$ admits a finite filtration whose associated gradeds are in $\mathcal{A}$. Then $i\colon \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ is an equivalence.*

"Devissage" is the word for "unscrewing" in French which was coined by Grothendieck in the K–theoretic setting. The graphic idea, as can perhaps be seen already in the statement above, is that we can "unscrew" or resolve, in a step–by–step manner, the K–theory of the bigger, more complicated category $\mathcal{B}$ and compute it as the K–theory of the smaller and simpler full subcategory $\mathcal{A}$. Another instance of such a "resolving" strategy is the following result.

Theorem 5.1.3 (Resolution theorem, Theorem 5.4.8)**.** *Let $\mathcal{C}$ be an exact category and let $\mathcal{U} \subseteq \mathcal{C}$ be a full subcategory satisfying that:*

(1) $\mathcal{U} \subseteq \mathcal{C}$ is closed under extensions;

(2) for every projection $p\colon u_1 \twoheadrightarrow u_2$ in $\mathcal{C}$ with $u_1, u_2 \in \mathcal{U}$, the fibre of p also lies in $\mathcal{U}$;

(3) for every $x \in \mathcal{C}$, there exists a finite length resolution

$$0 \to u_n \to \cdots \to u_2 \to u_1 \to u_0 \twoheadrightarrow x$$

where $u_i \in \mathcal{U}$ for all i.

Then the inclusion functor induces an equivalence $\mathrm{K}(\mathcal{U}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C})$.

We prove this by performing a devissage argument which reduces to proving the case where all objects admit a length 1 resolution, the case of which we quote from [Win24, Thm. 8.11].

The Theorems 5.1.1, 5.4.1 and 5.4.8 should already convince the reader of the value of defining K–theory for not necessarily stable categories, where classical homology computations may be ported and where the special method of devissage is at hand to further reduce the complexity of the problem. The next set of three results pertain to the orthogonal problem of relating the K–theory of stable categories to ones of the form where the above three results apply.

The first of this three is a theorem first proved by Neeman in the context of triangulated categories and then by Barwick in the higher categorical setting. The result -

which we prove in §5.6 - is formulated in terms of a notion called *t–structures* together with their associated *hearts*, which we introduce in §5.5. Simply put, it is an extra structure on stable categories that we often have, providing the stable analogue of the Postnikov filtration for anima.

Theorem 5.1.4 (Theorem of the t–heart, Theorem 5.6.10)**.** *Let $\mathcal{C}$ be a stable category equipped with a t–structure $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.*

(1) If the t–structure is bounded below, then both

$$\mathrm{K}(\mathcal{C}_{\geq 0}) \longrightarrow \mathrm{K}(\mathcal{C}) \qquad \mathrm{K}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathrm{K}(\mathcal{C}_{\leq 0})$$

are equivalences.

(2) If the t–structure is bounded, then $\mathrm{K}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathrm{K}(\mathcal{C})$ is an equivalence.

Combining Theorem 5.1.1, Theorem 5.1.2, and Theorem 5.1.4, we may then obtain the long–sought fibre sequence of spectra

$$\mathrm{k}(\mathbb{F}_p) \longrightarrow \mathrm{K}(\mathbb{Z}_{(p)}) \longrightarrow \mathrm{K}(\mathbb{Q})$$

of Quillen's that has been one of the guiding example of this course.

Finally, we mention two more result which we will *not* prove in this course. The first is an analogue of the Theorem of the t–heart in the setting of so–called *weight structures*, which may be thought of as the evil twin of a t–structure in that formally, it looks very similar to a t–structure, but in actuality it is quite a different gadget altogether.

Theorem 5.1.5 (Theorem of the weight heart, [Win24, Thm. 8.13])**.** *Let $\mathcal{C}$ be a stable category with a bounded weight structure. Then the inclusion of the weight heart induces an equivalence*

$$\mathrm{K}(\mathcal{C}^{\heartsuit_{\mathrm{w}}}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C})$$

This theorem has a tortuous history with many wrong proofs, and as far as we know, the first correct proof was given relatively recently in [HS23]. The last result, which we will not comment too much about, is the following:

Theorem 5.1.6 (Gillet–Waldhausen)**.** *Let $\mathcal{C}$ be a weakly idempotent–complete exact category. Then the map*

$$\mathrm{K}(j)\colon \mathrm{K}(\mathcal{C}) \longrightarrow \mathrm{K}(\mathcal{D}^{\mathrm{b}}(\mathcal{C}))$$

induced by the Gabriel–Quillen embedding $j\colon \mathcal{C} \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathcal{C})$ is an equivalence.

We refer the reader to [Win24, Thm. 8.13] and [Win24, Thm. 11.10] respectively, for a nice modern treatments of these results.

5.2 Waldhausen categories and the S–construction

Definition 5.2.1. A *Waldhausen category* is a category $\mathcal{C}$ together with a choice of subcategory $\mathrm{co}\mathcal{C}$ (whose morphisms we denote with $\rightarrowtail$ called *cofibrations*) such that:

(1) $\mathcal{C}$ is pointed and $0 \to x$ lies in $\mathrm{co}\mathcal{C}$ for all $x \in \mathcal{C}$;

(2) $\mathrm{co}\mathcal{C}$ contains $\mathcal{C}^{\simeq}$;

(3) every span $z \leftarrow x \to y$ with $x \rightarrowtail y$ in $\mathrm{co}\mathcal{C}$ admits a pushout and the induced map $z \to z \cup_x y$ also lies in $\mathrm{co}\mathcal{C}$.

An *exact functor of Waldhausen categories* is a functor of pairs $f\colon (\mathcal{C}, \mathrm{co}\mathcal{C}) \to (\mathcal{D}, \mathrm{co}\mathcal{D})$ which preserves zero objects and preserves pushouts along cofibrations. We write Wald for the category of small Waldhausen categories and exact functors.

We will often suppress mention of $\mathrm{co}\mathcal{C}$ and say that $\mathcal{C}$ is a Waldhausen category to mean that $(\mathcal{C}, \mathrm{co}\mathcal{C})$ is a Waldhausen category.

Example 5.2.2 (Split Waldhausen categories). A Waldhausen category $(\mathcal{C}, \mathrm{co}\mathcal{C})$ is said to be *split* if it has finite coproducts and if for all $x \rightarrowtail y$, the map $y \to z := 0 \cup_x y$ admits a section such that the induced map $x \sqcup z \to y$ is an equivalence. Since short exact sequences of projective modules are split, for a classical commutative ring R, Proj_R canonically yields a split Waldhausen category where the cofibrations are injections of projective modules whose cokernel is projective again. Note that while these are the maps which are summand inclusions, the decomposition into direct sums are *not* canonical!

Classically, the following notion was first defined by Quillen in [Qui73].

Definition 5.2.3 ([Win24, Def. 8.1])**.** An *exact category* is a category $\mathcal{C}$ together with two choices of subcategories $\mathrm{co}\mathcal{C}$, the *cofibrations* whose morphisms are denoted by $\rightarrowtail$, and $\mathrm{pr}\mathcal{C}$, the *projections* whose morphisms are denoted by $\twoheadrightarrow$, satisfying the following:

(1) $\mathcal{C}$ is additive;

(2) both $(\mathcal{C}, \mathrm{co}\mathcal{C})$ and $(\mathcal{C}^{\mathrm{op}}, (\mathrm{pr}\mathcal{C})^{\mathrm{op}})$ are Waldhausen categories;

(3) the following conditions on a commuting square

$$\begin{tikzcd} x \arrow[r, "i"] \arrow[d, "p"'] & y \arrow[d, "q"] \\ x' \arrow[r, "j"] & y' \end{tikzcd}$$

are equivalent:

- i is a cofibration, p is a projection, and the square is a pushout;
- j is a cofibration, q is a projection, and the square is a pullback.

An *exact functor* $F\colon (\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{pr}\mathcal{C}) \to (\mathcal{D}, \mathrm{co}\mathcal{D}, \mathrm{pr}\mathcal{D})$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that both $F\colon (\mathcal{C}, \mathrm{co}\mathcal{C}) \to (\mathcal{D}, \mathrm{co}\mathcal{D})$ and $F\colon (\mathcal{C}^{\mathrm{op}}, (\mathrm{pr}\mathcal{C})^{\mathrm{op}}) \to (\mathcal{D}^{\mathrm{op}}, (\mathrm{pr}\mathcal{D})^{\mathrm{op}})$ are exact functors of Waldhausen categories.

Surprisingly, the definition of an exact category is sufficiently "over–determined" that they may just be viewed as a Waldhausen category with some properties as encapsulated by the following:

Proposition 5.2.4 ([Win24, Cor. 8.6])**.** *The forgetful functor* $\mathrm{fgt}\colon \mathrm{Exact} \to \mathrm{Wald}$ *is fully faithful.*

Example 5.2.5. If $\mathcal{A}$ is an abelian category, then $\mathcal{A}$ acquires a canonical exact category structure by declaring the cofibrations to be the monomorphisms and projections to be the epimorphisms. Note that pushouts always exist in abelian categories since for maps $f\colon x \to y, g\colon x \to z$, we may compute the pushout $y \cup_x z$ as the cokernel of the map $x \xrightarrow{(f,-g)} y \oplus z$, and pullbacks also exist for similar reasons.

Example 5.2.6. If $\mathcal{C}$ is a stable category, then we may endow it with an exact category structure by declaring $\mathrm{co}\mathcal{C} = \mathcal{C}$ and $\mathrm{pr}\mathcal{C} = \mathcal{C}$.

Fact 5.2.7 ([Win24, Ex. 2.2(1)]). The category Wald is semiadditive, where the exact functors $\mathrm{id} \times 0\mathcal{C}_1 \to \mathcal{C}_1 \times \mathcal{C}_2$ and $0 \times \mathrm{id}\colon \mathcal{C}_2 \to \mathcal{C}_1 \times \mathcal{C}_2$ exhibits the product $\mathcal{C}_1 \times \mathcal{C}_2$ in Wald as a coproduct in Wald.

Observation 5.2.8. Note that the 1–category $\mathrm{Ar}[n] := \mathrm{Fun}(\Delta^1, [n])$ looks as follows:

$$\begin{tikzcd}
0,0 \arrow[r] & 0,1 \arrow[r] \arrow[d] & 0,2 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & 0,n \arrow[d] \\
 & 1,1 \arrow[r] & 1,2 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & 1,n \arrow[d] \\
 & & 2,2 \arrow[r] & \cdots \arrow[r] \arrow[d] & 2,n \arrow[d] \\
 & & & \vdots & \vdots \arrow[d] \\
 & & & & n,n
\end{tikzcd}$$

Construction 5.2.9 (Waldhausen's S–construction)**.** We construct a functor

$$\mathrm{S}_\bullet\colon \mathrm{Wald} \longrightarrow \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Wald})$$

known as *Waldhausen's* $\mathrm{S}_\bullet$*–construction.* For each $n \in \mathbb{N}$, the category $\mathrm{S}_n\mathcal{C}$ is defined to be the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$ on those objects $x\colon \mathrm{Ar}[n] \to \mathcal{C}$ satisfying that:

(1) $x_{i,i} \simeq 0$ for all i;

(2) $x_{i,j} \to x_{i,j+1}$ is a cofibration for all $0 \le i \le j \le n$;

(3) the square

$$\begin{tikzcd}
x_{i,j} \arrow[r, tail] \arrow[d] & x_{i,k} \arrow[d] \\
x_{j,j} \arrow[r, tail] & x_{j,k}
\end{tikzcd}$$

is a pushout for all $0 \le i \le j \le k \le n$.

For example, we have $\mathrm{S}_0\mathcal{C} \simeq *$, $\mathrm{S}_1\mathcal{C} \simeq \mathcal{C}$, and an object in $\mathrm{S}_2\mathcal{C}$ is given by a diagram

$$\begin{tikzcd}
0 \arrow[r, tail] & x_{0,1} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & x_{0,2} \arrow[d] \\
 & 0 \arrow[r, tail] & x_{1,2} \arrow[d] \\
 & & 0
\end{tikzcd}$$

where the square is a pushout. Note that since the second rows onwards may be computed by taking pushouts, in fact, $\mathrm{S}_n\mathcal{C}$ is equivalent to the full subcategory of $\mathrm{Fun}([n], \mathcal{C})$ consisting of all the sequences of cofibrations. In particular, we will often denote an object of $\mathrm{S}_n\mathcal{C}$ by $x_1 \rightarrowtail \cdots \rightarrowtail x_n$. We may endow $\mathrm{S}_n\mathcal{C}$ with a Waldhausen category structure by declaring transformations $f\colon x \to y$ to be cofibrations if the canonical map

$$x_{0,i+1} \cup_{x_{0,i}} y_{0,i} \longrightarrow y_{0,i+1}$$

is a cofibration for all $0 \le i < n$. It is a good exercise to check that this condition in particular implies that $f_{i,j}\colon x_{i,j} \to y_{i,j}$ is a cofibration for all $0 \le i \le j \le n$.

The advantage of using $\mathrm{Ar}[n]$ instead of $[n]$ in the definition of $\mathrm{S}_\bullet\mathcal{C}$ is that it makes clear that $\mathrm{S}_\bullet\mathcal{C}$ assembles to a simplicial object in Wald. Working out the face maps explicitly, we get that the face map $d_0\colon \mathrm{S}_n\mathcal{C} \rightarrowtail \mathrm{S}_{n-1}\mathcal{C}$ is given by forgetting the top row (i.e. forgetting the terms $x_{0,i}$), and for $j > 0$, $d_j\colon \mathrm{S}_n\mathcal{C} \to \mathrm{S}_{n-1}\mathcal{C}$ is given by forgetting the $x_{\bullet,j}$ column. In the special case of $\mathrm{S}_2\mathcal{C}$, we will also use the special notations

$$q \coloneqq d_0,\ t \coloneqq d_1,\ s \coloneqq d_2\colon \mathrm{S}_2\mathcal{C} \longrightarrow \mathcal{C}$$

since these operations precisely corresponds to taking the quotient, target, and source of a cofibration $x_{0,1} \rightarrowtail x_{0,2}$.

Lemma 5.2.10. *Let* $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$. *Then* $\pi_0|X_\bullet| \cong \pi_0 X_0/[d_0 \sim d_1]$.

Proof. Since the solid square in

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) \arrow[r, dashed, shift left, "|-|"] \arrow[d, dashed, shift right, "\pi_0"'] & \mathrm{An} \arrow[l, shift left, "\mathrm{const}_\bullet"] \arrow[d, dashed, shift right, "\pi_0"'] \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set}) \arrow[r, dashed, shift left, "|-|"] \arrow[u, hook, shift right] & \mathrm{Set} \arrow[l, shift left, "\mathrm{const}_\bullet"] \arrow[u, hook, shift right]
\end{tikzcd}$$

clearly commutes, so does the dashed square of left adjoints. The claimed result is then an immediate consequence of the coequaliser formula in sets. □

Observation 5.2.11. First of all, note that since Wald is semiadditive, the functor $|\mathrm{S}_\bullet(-)^{\simeq}|\colon \mathrm{Wald} \to \mathrm{An}$ clearly preserves finite products, and so it factors through CMon. Moreover, since $\mathrm{S}_0\mathcal{A} \simeq *$, we obtain from Lemma 5.2.10 that $\pi_0|(\mathrm{S}_\bullet\mathcal{A})^{\simeq}| \simeq *$, and so $|(\mathrm{S}_\bullet\mathcal{A})^{\simeq}|$ is even group–like. Thus, we have the following canonical lift

$$|\mathrm{S}_\bullet(-)^{\simeq}|\colon \mathrm{Wald} \longrightarrow \mathrm{CGrp}$$

We now argue that it preserves filtered colimits. This is because $\mathrm{Fun}([n], -)$ preserves filtered colimits and one can check that this implies that $\mathrm{S}_n(-)$ also does. Moreover, $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{An}$ preserves filtered colimits since the left adjoint $\mathrm{incl}\colon \mathrm{An} \hookrightarrow \mathrm{Cat}$ does because it preserves compact objects (which are finite colimits and retracts of $* = \Delta^0$). Finally, the forgetful functor $\mathrm{fgt}\colon \mathrm{CMon} \to \mathrm{An}$ preserves filtered colimits, so we do indeed get the desired claim.

Observation 5.2.12. Note that the notion of an exact category is self–dual in that if $(\mathcal{C}, \mathrm{co}\mathcal{C}, \mathrm{pr}\mathcal{C})$ is an exact category, then the same is true for $(\mathcal{C}^{\mathrm{op}}, (\mathrm{pr}\mathcal{C})^{\mathrm{op}}, (\mathrm{co}\mathcal{C})^{\mathrm{op}})$. Moreover, one can check that

$$\mathrm{S}_\bullet\mathcal{C} \simeq \mathrm{S}_\bullet(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$$

In particular, we see that $\mathrm{K}(\mathcal{C}) \simeq \mathrm{K}(\mathcal{C}^{\mathrm{op}})$.

Definition 5.2.13. Let $\mathcal{A}$ be a category with finite products. A functor $H\colon \mathrm{Wald} \to \mathcal{A}$ is said to be:

- *reduced* if $H(0) \simeq *$;
- *finitary* if it preserves filtered colimits;
- *grouplike* if the shear functor of Waldhausen categories $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ induces an equivalence $H(\mathcal{C} \times \mathcal{C}) \xrightarrow{\simeq} H(\mathcal{C} \times \mathcal{C})$.

An *additive invariant* is a finitary, reduced, and grouplike functor $H\colon \mathrm{Wald} \to \mathcal{A}$ such that the map

$$(Hs, Hq)\colon H\mathrm{S}_2\mathcal{C} \longrightarrow H\mathcal{C} \times H\mathcal{C}$$

is an equivalence. We write $\mathrm{Fun}_*(\mathrm{Wald}, \mathcal{A})$ and $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Wald}, \mathcal{A})$ for the full subcategories of reduced and additive functors, respectively.

Exercise 5.2.14 ([Win24, Rmk. 1.12])**.** We would like to argue that additive invariants are automatically product–preserving. So let $H\colon \mathrm{Wald} \to \mathcal{A}$ be an additive invariant.

(a) Show that the exact functor

$$e\colon \mathcal{C} \times \mathcal{C} \to \mathrm{S}_2\mathcal{C} \quad :: \quad (x, y) \mapsto (x \rightarrowtail x \sqcup y)$$

defines a section to the map $(s, q)\colon \mathrm{S}_2\mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

(b) Thus, show that the composite of any two consecutive arrows in the sequence

$$H(\mathcal{C}\times\mathcal{C}) \xrightarrow{He} H\mathrm{S}_2\mathcal{C} \xrightarrow{H(s,q)} H(\mathcal{C}\times\mathcal{C}) \longrightarrow H\mathcal{C}\times H\mathcal{C}$$

is an equivalence and deduce that the last map is an equivalence.

(c) Conclude that we have a natural equivalence

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Wald},\mathrm{CGrp}(\mathcal{A}))\simeq\mathrm{Fun}^{\mathrm{add}}(\mathrm{Wald},\mathcal{A}).$$

Theorem 5.2.15 (Barwick, Waldhausen, [Bar16, Thm. 7.9, Cor. 7.14])**.** *The inclusion* $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Wald},\mathrm{An})\hookrightarrow\mathrm{Fun}_*(\mathrm{Wald},\mathrm{An})$ *admits a left adjoint given by* $\Omega|\mathrm{S}_\bullet(-)^{\simeq}|$.

To end this section, we record here that the Q–construction and S–construction both give rise to the same thing. The following result was first proved by Waldhausen, and we refer also to [HW21, Prop. IV.8] and [Win24, §12.5] for more recent write–ups. We will freely use this result from now on.

Theorem 5.2.16 (S=Q, [Wal85, App. 1.9])**.** *There is a natural equivalence* $|\mathrm{S}_\bullet(-)^{\simeq}|\Rightarrow|\mathrm{Q}_\bullet(-)^{\simeq}|$ *of functors* $\mathrm{Exact}\to\mathrm{CGrp}$.

While we will neither repeat the proof nor the construction of the transformation here, let us mention that the point of it is that one may “find” the Q–construction inside the S–construction by using the edgewise subdivision functor. For example, within a diagram in $\mathrm{S}_5\mathcal{C}$

$$\begin{tikzcd}
x_{01} \arrow[r, tail] & x_{02} \arrow[r, tail] \arrow[d, two heads] & \color{red}x_{03} \arrow[r, tail, red] \arrow[d, two heads, red] & \color{red}x_{04} \arrow[r, tail, red] \arrow[d, two heads, red] & \color{red}x_{05} \arrow[d, two heads] \\
 & x_{12} \arrow[r, tail] & \color{red}x_{13} \arrow[r, tail, red] \arrow[d, two heads, red] & \color{red}x_{14} \arrow[r, tail] \arrow[d, two heads] & x_{15} \arrow[d, two heads] \\
 & & \color{red}x_{23} \arrow[r, tail] & x_{24} \arrow[r, tail] \arrow[d, two heads] & x_{25} \arrow[d, two heads] \\
 & & & x_{34} \arrow[r, tail] & x_{35} \arrow[d, two heads] \\
 & & & & x_{45}
\end{tikzcd}$$

we may find a diagram in $\mathrm{Q}_2\mathcal{C}$ given by the red portion.

5.3 The +=S theorem

Notation 5.3.1. In this section, we will use the following notations

$$\begin{tikzcd}
\Delta^{\mathrm{op}}_{\leq 1} \arrow[r, hook, "i"] \arrow[dr, "q"'] & \Delta^{\mathrm{op}} \arrow[d, "q"] \\
 & *
\end{tikzcd}$$

for the canonical maps. Notice that, for $\mathcal{A} \in \mathrm{Wald}$, $i^*\mathrm{S}_\bullet\mathcal{A}$ is precisely given by the diagram $(0 \leftarrow \mathcal{A} \to 0)$, which is the diagram defining pushouts. In particular, we get

$$|i_!i^*(\mathrm{S}_\bullet\mathcal{A})^{\simeq}| \simeq q_!i^*(\mathrm{S}_\bullet\mathcal{A})^{\simeq} \simeq \Sigma\mathcal{A}^{\simeq} \in \mathrm{CMon}$$

Here, it is important that we take the left Kan extension $i_!i^*(\mathrm{S}_\bullet\mathcal{A})^{\simeq}$ in CMon and not in An.

Construction 5.3.2 (Simplicial suspensions)**.** We define $\Sigma_\bullet\mathcal{A} := i_!i^*\mathrm{S}_\bullet\mathcal{A} \in \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{Wald})$. We thank God that the combinatorics has been done in [HW21, Pf. of Prop. II.16]. By the analysis of the slice categories $(\mathbb{\Delta}^{\mathrm{op}}_{\leq 1})_{/[n]}$ in loc. cit. and by the left Kan extension formula, we know that for all n, we have

$$\big(i_!i^*\mathrm{S}_\bullet\mathcal{A}\big)([n]) \simeq \operatorname{colim}\Big((\mathbb{\Delta}^{\mathrm{op}}_{\leq 1})_{/[n]} \to \mathbb{\Delta}^{\mathrm{op}}_{\leq 1} \xrightarrow{i^*\mathrm{S}_\bullet\mathcal{A}} \mathrm{CMon}\Big) \simeq \mathcal{A}^{\oplus n}$$

where the i–th copy of $\mathcal{A}$ in $\mathcal{A}^{\oplus n}$ corresponds to the surjection $s_i\colon [n] \twoheadrightarrow [1]$ represented by $00\cdots 011\cdots 1$ where 1 starts at the i–th coordinate (counting with the first term being the 0–th coordinate). Moreover, this also means that the adjunction counit map

$$\big(\Sigma_\bullet\mathcal{A}\big)([n]) = \big(i_!i^*\mathrm{S}_\bullet\mathcal{A}\big)([n]) \simeq \mathcal{A}^{\oplus n} \longrightarrow \big(\mathrm{S}_\bullet\mathcal{A}\big)([n]) = \mathrm{S}_n\mathcal{A} \tag{5.1}$$

is given concretely as follows: for the i–th summand in $\mathcal{A}^{\oplus n}$, the map $\mathcal{A} \to \mathrm{Fun}([n], \mathcal{A})$ is given by the map s_i in the simplicial structure on $\mathrm{S}_\bullet\mathcal{A}$, i.e. it sends $a \in \mathcal{A}$ to the diagram in $\mathrm{S}_n\mathcal{A}$ extended from $(0 = 0\cdots = 0 \to a = a\cdots = a) \in \mathrm{Fun}([n], \mathcal{A})$. Thus, in total, the map (5.1) sends $(a_1, \ldots, a_n)$ to the diagram in $\mathrm{S}_n\mathcal{A}$ extended from

$$\big(a_1 \to a_1 \oplus a_2 \to a_1 \oplus a_2 \oplus a_3 \to \cdots \to a_1 \oplus \cdots \oplus a_n\big) \in \mathrm{Fun}(\Delta^n, \mathcal{A})$$

Using for example that the map (5.1) is a map of simplicial objects, we may use the descriptions of the face maps on $\mathrm{S}_\bullet\mathcal{A}$ to obtain the following descriptions of the face maps on $\Sigma_\bullet\mathcal{A}$: for the map $d_i\colon [n-1] \to [n]$ in $\mathbb{\Delta}$, we have:

$$\begin{array}{llll} d_0\colon & (a_1, \ldots, a_n) & \mapsto & (a_2, \ldots, a_n) \\ d_1\colon & (a_1, \ldots, a_n) & \mapsto & (a_1 \oplus a_2, \ldots, a_n) \\ \vdots & \vdots & \vdots & \vdots \\ d_{n-1}\colon & (a_1, \ldots, a_n) & \mapsto & (a_1, \ldots, a_{n-1} \oplus a_n) \\ d_n\colon & (a_1, \ldots, a_n) & \mapsto & (a_1, \ldots, a_{n-1}) \end{array}$$

Our goal in this section is to prove Theorem 5.3.17 which says that when $\mathcal{A}$ is a split Waldhausen category (c.f. Example 5.2.2), the map

$$\Omega|(\Sigma_\bullet\mathcal{A})^{\simeq}| \longrightarrow \Omega|(\mathrm{S}_\bullet\mathcal{A})^{\simeq}|$$

induced by the adjunction counit is an equivalence. This would say that the group–completion K–theory of $\mathcal{A}$ is the same as the Waldhausen K–theory of $\mathcal{A}$. To put it in a standard slogan, this says that "+=Q" (or rather, +=S).

This is a rather intuitive fact when one considers the following: let us work with $\bullet = 2$ for simplicity. Then the map is given by

$$\Sigma_2\mathcal{A} = \mathcal{A}^{\oplus 2} \longrightarrow \mathrm{S}_2\mathcal{A} \quad :: \quad (a,b) \mapsto (a \rightarrowtail a \oplus b)$$

Since $\mathcal{A}$ was split, all objects in $\mathrm{S}_2\mathcal{A}$ will look like the image of this map.

However, this map is not fully faithful since maps between $(a \rightarrowtail a \oplus b)$ and $(a' \rightarrowtail a' \oplus b')$ might not be induced by maps $(a,b) \to (a',b')$ in $\mathcal{A}^{\oplus 2}$. One might try to build an adjoint map $\mathrm{S}_2\mathcal{A} \to \mathcal{A}^{\oplus 2}$ and then invoke that $|-|$ sends adjunctions to equivalences. However, this fails since splittings are not natural which might obstruct us from building the requisitie adjunction (co)units, and in any case, we have to first apply $(-)^{\simeq}$ to the maps in question, which destroys the putative adjunctions since the (co)units might not be equivalences. Therefore, we will have to work quite a bit harder and think quite a bit smarter to prove the desired result, and this was what Waldhausen did in [Wal85, §1.8]. We now present his solution.

The basic strategy is quite a natural one: when one is faced with the problem of showing that a functor becomes an equivalence upon geometric realisation and where building a naturally expected adjunction fails, one tries to use Quillen's Theorem A by performing "local" computations. Our situation is not quite the generic one though, so we will need a modified form of the standard theorem.

Observation 5.3.3. Note that the Rezk nerve $N_\bullet([n]) \in s\mathrm{An}$ of the categories $[n] \in \mathrm{Cat}$ have the form $[m] \mapsto \coprod_{[m]\to[n]} *$ where the coproduct runs over all maps $[m] \to [n]$ in $\Delta \subseteq \mathrm{Cat}$. This is simply because $\mathrm{Map}_{\mathrm{Cat}}([m],[n])$ is discrete.

Exercise 5.3.4. Let $J \in \mathrm{Cat}$. Recall that by the *cartesian unstraightening*, we mean the following composite

$$\mathrm{UnStr}c\colon\ \mathrm{Fun}(J,\mathrm{Cat}) \xrightarrow[\simeq]{(-)^{\mathrm{op}}} \mathrm{Fun}(J,\mathrm{Cat}) \simeq \mathrm{coCart}(J) \xrightarrow[\simeq]{(-)^{\mathrm{op}}} \mathrm{Cart}(J^{\mathrm{op}}).$$

Let $A_\bullet \in \mathrm{Fun}(J,\mathrm{Cat})$. Show using Theorem 2.1.41 and that $|-| \simeq |(-)^{\mathrm{op}}|\colon \mathrm{Cat} \to \mathrm{An}$ that we may also compute $|\operatorname{colim}_J A_\bullet| \in \mathrm{An}$ as $|\mathrm{UnStr}c(A_\bullet)|$.

Proposition 5.3.5 (Simplicial Quillen's Theorem A, [Wal82, §4])**.** *Suppose we have a map of simplicial categories $p\colon \mathcal{C}_\bullet \to \mathrm{const}_\bullet\mathcal{B}$. If for all $B \in \mathcal{B}$, $|p_\bullet \downarrow B| \simeq *$, then $|p|\colon |\mathcal{C}_\bullet| \to |\mathrm{const}_\bullet\mathcal{B}| \simeq |\mathcal{B}|$ is an equivalence in anima.*

Proof. Since $|-| \simeq |(-)^{\mathrm{op}}|$, we may also take the cartesian unstraightening and use Theorem 2.1.41 to compute $|-|$. So let $\mathrm{UnStr}c(\mathcal{C}_\bullet), \mathrm{UnStr}c(\mathrm{const}_\bullet\mathcal{B}) \simeq \Delta\times\mathcal{B} \in \mathrm{Cart}_{/\Delta}$. By Quillen's Theorem A, if we can show that for all $([n],B) \in \mathrm{UnStr}c(\mathrm{const}_\bullet\mathcal{B})$, $|\mathrm{UnStr}c(p) \downarrow ([n],B)| \simeq *$, then we would have shown that $|\mathrm{UnStr}c(p)| \simeq |p|$ is an equivalence.

To this end, first observe that from the pullback construction of $(\mathrm{UnStr}c(p) \downarrow ([n],B))$

$$\begin{tikzcd}
\big(\mathrm{UnStr}c(p) \downarrow ([n], B)\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{UnStr}c(\mathrm{const}_\bullet \mathcal{B})_{/([n],B)} \arrow[d] \\
\mathrm{UnStr}c(\mathcal{C}_\bullet) \arrow[r] & \mathrm{UnStr}c(\mathrm{const}_\bullet \mathcal{B}),
\end{tikzcd}$$

we obtain a functor $q\colon (\mathrm{UnStr}c(p) \downarrow ([n], B)) \to \mathrm{UnStr}c(\mathcal{C}_\bullet) \to \Delta$ via the left vertical map. Note that the right vertical map is a cartesian fibration and cartesian fibations are preserved under pullbacks, hence the left vertical map is cartesian too. Since cartesian fibrations are closed under compositions, we get that $q\colon (\mathrm{UnStr}c(p) \downarrow ([n], B)) \to \Delta$ is a cartesian fibration. Thus, by cartesian straightening, we may view $(\mathrm{UnStr}c(p) \downarrow ([n], B))$ as a simplicial category, and we claim that it is given by $N_\bullet([n]) \times (p_\bullet \downarrow B)$ where $N_\bullet([n])$ is the Rezk nerve of $[n]$.

To see this, we only have to compute the fibre of $(\mathrm{UnStr}c(p) \downarrow ([n], B))$ over each $[m] \in \Delta$, since the simplicial structures will then be clear by construction. So fix an $[m] \in \Delta$. Note that $([n], B) \simeq s^*([0], B)$ where $s\colon [n] \to [0]$ is the unique map in Δ, using that $[0] \in \Delta$ is terminal. Write also $v\colon [m] \to [0]$ for the unique map. Therefore, we obtain the identifications

$$(\mathrm{UnStr}c(p) \downarrow ([n], B)) \simeq \coprod_{u\colon [m]\to[n]} (p_m \downarrow u^* s^*([0], B)) \simeq \Big(\coprod_{u\colon [m]\to[n]} *\Big) \times (p_m \downarrow v^*([0], B))$$

where the first equivalence is by virtue of the fact that q was a cartesian fibration over Δ. Thus, by the description of the Rezk nerve of $[n]$ from Observation 5.3.3, we obtain the claim as desired. To complete the proof, simply note that $|\mathrm{UnStr}c(p) \downarrow ([n], B)| \simeq |N_\bullet([n])| \times |p_\bullet \downarrow B| \simeq |p_\bullet \downarrow B| \simeq *$ by hypothesis. □

Our main tool for the proof will be the relative version of Construction 5.3.2, akin to the relative Q–construction used in the proof of Theorem 4.4.6, whose purpose is to fit the geometric realisation of a simplicial functor into a fibre sequence.

Construction 5.3.6 (Relative suspension construction, [Wal85, p. 370])**.** Let $f\colon \mathcal{A} \to \mathcal{B}$ be a coproduct–preserving functor between categories with finite coproducts. We define the simplicial object $\Sigma_\bullet f$ as the pullback

$$\begin{tikzcd}
\Sigma_\bullet f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D\acute{e}c}\Sigma_\bullet\mathcal{B} \arrow[d, "\partial_0"] \\
\Sigma_\bullet\mathcal{A} \arrow[r, "f"] & \Sigma_\bullet\mathcal{B}
\end{tikzcd} \tag{5.2}$$

Since $\Sigma_n\mathcal{B} \simeq \mathcal{B}^{\times n}$ and $(\mathrm{D\acute{e}c}\Sigma_\bullet\mathcal{B})_n \simeq \Sigma_{n+1}\mathcal{B} \simeq \mathcal{B}^{\times n+1}$, we can check that the right vertical map ∂_0 is projecting away from the first coordinate so that concretely, $\Sigma_n f \simeq \mathcal{B} \times \mathcal{A}^{\times n}$. From this, we get a sequence of simplicial objects

$$\mathrm{const}_\bullet \mathcal{B} \longrightarrow \Sigma_\bullet f \longrightarrow \Sigma_\bullet\mathcal{A} \tag{5.3}$$

where the first map is obtained using the pullback property of $\Sigma_\bullet f$ and that the right adjoint of the functor $\mathrm{const}_\bullet$ is ev_0.

Remark 5.3.7. One might be slightly confused why this is a valid analogue of the relative Q–construction from the proof of Theorem 4.4.6, where the upper right term was called Null instead. The point here is that the Null construction is basically the décalage construction with only a minor tweak that one term in the diagram is zero.

We really want to say that (5.3) is a fibre sequence upon geometric realisation, but it is not in general... The next two results give a fix to this fibration pipe dream by suspending once.

Lemma 5.3.8. *Let* $f\colon M \to G$ *be a map in* CMon *where* G *is group–like. Then the pullback square in* $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{inj}}, \mathrm{An})$

$$\begin{tikzcd}
\Sigma_\bullet f \arrow[r] \arrow[d] & \mathrm{D\acute{e}c}\Sigma_\bullet G \arrow[d, "\partial_0"] \\
\Sigma_\bullet M \arrow[r, "f"] & \Sigma_\bullet G
\end{tikzcd}$$

of semisimplicial objects is an equifibred square.

Proof. We have to verify that the map $\partial_0\colon \mathrm{D\acute{e}c}\Sigma_\bullet G \to \Sigma_\bullet G$ is equifibred. It would suffice to prove this for every $d_i\colon [n-1] \to [n]$ in Δ. In most cases the squares involved will just be projections, in which case they are clearly going to be cartesian. The interesting case is for $i = 0$. In this case, the square to consider is the left square in

$$\begin{tikzcd}
G^{\times(n+1)} \arrow[r, "d_0^{\mathrm{D\acute{e}c}}=d_1=(\oplus{,}\mathrm{id}{,}\dots{,}\mathrm{id})"] \arrow[d, "\partial_0"'] & G^{\times n} \arrow[d, "\partial_0"] & G^{\times 2} \arrow[r, "\oplus"] \arrow[d, "\mathrm{proj}_1"'] & G \arrow[d] \\
G^{\times n} \arrow[r, "d_0"] & G^{\times(n-1)} & G \arrow[r] & 0
\end{tikzcd}$$

After taking away the irrelevant factors, we are reduced to showing that the square on the right is a pullback. This is so since the induced map from $G^{\times 2}$ to the pullback is precisely the shear map $G^{\times 2} \to G^{\times 2}$, which is an equivalence by hypothesis that G was group–like. □

Notice that there is a natural equivalence of bisimplicial objects $\Sigma_\bullet(\mathrm{S}_\bullet f) \simeq \mathrm{S}_\bullet(\Sigma_\bullet f)$.

Proposition 5.3.9 (Waldhausen's suspended fibration, [Wal85, Lem. 1.8.6])**.** *Let* $f\colon \mathcal{A} \to \mathcal{B}$ *be an exact functor of Waldhausen categories. The sequences*

$$|(\Sigma_\bullet\mathcal{B})^\simeq| \to |(\Sigma_\bullet\Sigma_\bullet f)^\simeq| \to |(\Sigma_\bullet\Sigma_\bullet\mathcal{A})^\simeq| \qquad |(\mathrm{S}_\bullet\mathcal{B})^\simeq| \to |(\mathrm{S}_\bullet\Sigma_\bullet f)^\simeq| \to |(\mathrm{S}_\bullet\Sigma_\bullet\mathcal{A})^\simeq|$$

are fibre sequences in CGrp*. Moreover, if* f *was an equivalence, then the middle terms are contractible.*

Proof. We prove only the second fibre sequence, since the first is similar and slightly easier. Since we are taking geometric realisations, we may as well consider the associated semisimplicial diagrams. We claim that the pullback in $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{inj}}, \mathrm{An})$

$$\begin{tikzcd}
\Sigma_\bullet|\mathrm{S}_\bullet f| \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D\acute{e}c}\Sigma_\bullet|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}| \arrow[d] \\
\Sigma_\bullet|(\mathrm{S}_\bullet\mathcal{A})^{\simeq}| \arrow[r] & \Sigma_\bullet|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|
\end{tikzcd}$$

is equifibred. By Observation 5.2.11, we know that $|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|$ is a group–like $\mathbb{E}_\infty$–monoid anima. Hence, by Lemma 5.3.8, the square is indeed equifibred as claimed. Since $|\mathrm{D\acute{e}c}\Sigma_\bullet|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|| \simeq *$, the right vertical fibre is $\Omega|\Sigma_\bullet|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|| \simeq \Omega\Sigma|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}| \simeq |(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|$, where the second equivalence is again by using that $|(\mathrm{S}_\bullet\mathcal{B})^{\simeq}|$ is group–like. Thus, extracting the left vertical fibre yields the required fibre sequence. □

Our next aim is to show Proposition 5.3.15, which is a very general form of Theorem 5.3.17. To this end, we begin our analysis of the requisite "local" considerations.

Construction 5.3.10. Let $\mathcal{C}$ be a Waldhausen category with finite coproducts and pushouts. For $X \in \mathcal{C}$, we define $\mathcal{C}_X$ to be the category whose objects are cofibrations $(X \rightarrowtail A)$. This category admits coproducts given by pushouts in $\mathcal{C}$. We then get a functor $j\colon \mathcal{C} \to \mathcal{C}_X$ given by $A \mapsto X \oplus A$. Note that this is coproduct–preserving. However, beware that in general $\mathcal{C}_X$ can no longer be a Waldhausen category since it is not even pointed.

Lemma 5.3.11. *Let $\mathcal{C}$ be an additive category with pushouts and $X \in \mathcal{C}$. Then there is a natural equivalence of functors $\mathcal{C}_X \to \mathcal{C}_X$ given by $(X \rightarrowtail A) \mapsto (X \rightarrowtail A \cup_X A)$ and $(X \rightarrowtail A) \mapsto (X \rightarrowtail A \oplus A/X)$.*

Proof. Since $\mathcal{C}$ was additive, we have a natural equivalence of functors $\mathcal{C} \to \mathcal{C}$ given by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} : A \oplus A \xleftrightarrow{\simeq} A \oplus A : \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

Now consider the map of cofibre sequences

$$\begin{tikzcd}
X \arrow[r, "(-i\ i)"] \arrow[d, equal] & A \oplus A \arrow[r] \arrow[d, "\simeq", "\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}"'] & A \cup_X A \arrow[d] \\
X \arrow[r, "(0\ i)"'] & A \oplus A \arrow[r] & A \oplus A/X
\end{tikzcd}$$

Since the left and middle vertical maps are equivalences, so is the right vertical map, as was to be shown. □

Proposition 5.3.12. *Let $\mathcal{C}$ be additive and $X \in \mathcal{C}$. Then the coproduct–preserving map $j\colon \mathcal{C} \to \mathcal{C}_X$ induces an equivalence in* CGrp

$$|(\Sigma_\bullet\mathcal{C})^{\simeq}| \longrightarrow |(\Sigma_\bullet\mathcal{C}_X)^{\simeq}|$$

Proof. Since the forgetful functor $\mathrm{fgt}\colon \mathrm{CGrp} \to \mathrm{An}$ is conservative, it would suffice to show that this is an equivalence of anima. Write the cofibre functor as $q\colon \mathcal{C}_X \to \mathcal{C} \;::\; (X \rightarrowtail A) \mapsto A/X$. This might not necessarily be a coproduct–preserving functor, but it would not matter for our purposes. Since the composition $\mathcal{C} \xrightarrow{j} \mathcal{C}_X \xrightarrow{q} \mathcal{C}$ is equivalent to the identity, it would suffice to now to show that the composite $jq\colon |(\Sigma_\bullet \mathcal{C}_X)^{\simeq}| \to |(\Sigma_\bullet \mathcal{C}_X)^{\simeq}|$ is also equivalent to the identity. Now, by Lemma 5.3.11, we have an equivalence of maps

$$\mathrm{id} \vee \mathrm{id} \simeq \mathrm{id} \vee jq\colon |(\Sigma_\bullet \mathcal{C}_X)^{\simeq}| \longrightarrow |(\Sigma_\bullet \mathcal{C}_X)^{\simeq}|$$

But since $|(\Sigma_\bullet \mathcal{C}_X)^{\simeq}| \simeq \Sigma \mathcal{C}_X^{\simeq} \in \mathrm{CMon}$ is group–like, we may subtract id from both sides to get $\mathrm{id} \simeq jq$, as wanted. □

Construction 5.3.13. Let $j_n\colon \mathcal{D} \to \mathrm{S}_n\mathcal{D}$ be the map given by $X \mapsto (0 \rightarrowtail 0 \rightarrowtail \cdots \rightarrowtail 0 \rightarrowtail X)$. This is one of the degeneracy maps in the simplicial structure of $\mathrm{S}_\bullet\mathcal{D}$. Note that there is also a map (which is one of the simplicial face maps)

$$\mathrm{S}_n\mathcal{D} \longrightarrow \mathrm{S}_{n-1}\mathcal{D} \quad :: \quad (A_1 \rightarrowtail \cdots \rightarrowtail A_n) \mapsto (A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1})$$

From this, we obtain a map

$$p\colon |\Sigma_\bullet(j_n\colon \mathcal{D} \to \mathrm{S}_n\mathcal{D})^{\simeq}| \longrightarrow |\,\mathrm{const}_\bullet(\mathrm{S}_{n-1}\mathcal{D})^{\simeq}| \simeq |(\mathrm{S}_{n-1}\mathcal{D})^{\simeq}|$$

The following is the key computation for our local analysis using Proposition 5.3.5, which should also explain why the maps $j\colon \mathcal{D} \to \mathcal{D}_X$ is relevant to our problem at all.

Lemma 5.3.14 ([Wal85, Lem. 1.8.8])**.** *Let $\mathcal{D} \in \mathrm{Wald}$ admitting pushouts. If for every $X \in \mathcal{D}$, the anima $|\Sigma_\bullet(j\colon \mathcal{D} \to \mathcal{D}_X)^{\simeq}|$ is contractible, then the map $p\colon |\Sigma_\bullet(j_n\colon \mathcal{D} \to \mathrm{S}_n\mathcal{D})^{\simeq}| \to |(\mathrm{S}_{n-1}\mathcal{D})^{\simeq}|$ is an equivalence.*

Proof. By Proposition 5.3.5, we have to show for every $\underline{B} = (B_1 \rightarrowtail \cdots \rightarrowtail B_{n-1}) \in (\mathrm{S}_{n-1}\mathcal{D})^{\simeq}$ that $|p_\bullet \downarrow \underline{B}| \simeq *$. At level m, the map $p\colon (\mathrm{S}_n\mathcal{D} \times \mathcal{D}^{\times m})^{\simeq} \to \mathrm{S}_{n-1}\mathcal{D}^{\simeq}$ is given by

$$(A_1 \rightarrowtail \cdots \rightarrowtail A_n; D_1, \ldots, D_m) \mapsto (A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1}).$$

Hence, objects in the anima $(p_m \downarrow \underline{B})$ are given by a diagram

$$\begin{tikzcd}
A_1 \arrow[r, tail] \arrow[d, "\simeq"] & \cdots \arrow[r, tail] & A_{n-1} \arrow[r, tail] \arrow[d, "\simeq"] & A_n \\
B_1 \arrow[r, tail] & \cdots \arrow[r, tail] & B_{n-1} &
\end{tikzcd} \tag{5.4}$$

together with the datum of an m–tuple $(D_1, \ldots, D_m) \in \mathcal{D}^{\times m}$. Observe that every object in $(p_m \downarrow \underline{B})$ is equivalent to one where the vertical maps in (5.4) are identities. Hence, since for these objects, the data $A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1}$ are fixed to be $B_1 \rightarrowtail \cdots \rightarrowtail B_{n-1}$, we obtain that $(p_m \downarrow \underline{B}) \simeq (\mathcal{D}_{B_{n-1}} \times \mathcal{D}^{\times m})^{\simeq}$ and thus that $(p_\bullet \downarrow \underline{B}) \simeq \Sigma_\bullet(\mathcal{D} \to \mathcal{D}_{B_{n-1}})^{\simeq} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$. Therefore, by our hypothesis, we get that $|p_\bullet \downarrow \underline{B}| \simeq *$, as required. □

We are now ready to prove the abstract version of +=S.

Proposition 5.3.15 (Abstract +=S, [Wal85, Prop. 1.8.7])**.** *Let* $\mathcal{D} \in \mathrm{Wald}$ *admitting pushouts. If for every* $X \in \mathcal{D}$, $|\Sigma_\bullet(j\colon \mathcal{D} \to \mathcal{D}_X)^{\simeq}|$ *is contractible, then the map* $|(\Sigma_\bullet \mathcal{D})^{\simeq}| \to |(\mathrm{S}_\bullet \mathcal{D})^{\simeq}|$ *is an equivalence.*

Proof. By Proposition 5.3.9 applied to the exact functor $\mathrm{id}\colon \mathcal{D} \xrightarrow{=} \mathcal{D}$, it suffices to show that $|(\Sigma_\bullet\Sigma_\bullet\mathcal{D})^{\simeq}| \to |(\Sigma_\bullet \mathrm{S}_\bullet\mathcal{D})^{\simeq}|$ is an equivalence, and for this, it suffices to show that $|(\Sigma_\bullet\Sigma_n\mathcal{D})^{\simeq}| \to |(\Sigma_\bullet \mathrm{S}_n\mathcal{D})^{\simeq}|$ is an equivalence for every n. By induction on n, it thus suffices to show that the map

$$|(\Sigma_\bullet \mathrm{S}_n\mathcal{D})^{\simeq}| \longrightarrow |(\Sigma_\bullet \mathrm{S}_{n-1}\mathcal{D} \times \Sigma_\bullet\mathcal{D})^{\simeq}|$$

induced by the functor $\mathrm{S}_n\mathcal{D} \to \mathrm{S}_{n-1}\mathcal{D} \times \mathcal{D}$ given by $(A_1 \rightarrowtail \cdots \rightarrowtail A_n) \mapsto (A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1}, A_n/A_{n-1})$ is an equivalence. But by Proposition 5.3.9 again, we have a map of fibre sequences

$$\begin{tikzcd}
{|(\Sigma_\bullet \mathrm{S}_n\mathcal{D})^{\simeq}|} \arrow[r] \arrow[d] & {|(\Sigma_\bullet\Sigma_\bullet(j_n\colon \mathcal{D} \to \mathrm{S}_n\mathcal{D}))^{\simeq}|} \arrow[r] \arrow[d] & {|(\Sigma_\bullet\Sigma_\bullet\mathcal{D})^{\simeq}|} \arrow[d, equal] \\
{|(\Sigma_\bullet \mathrm{S}_{n-1}\mathcal{D} \times \Sigma_\bullet\mathcal{D})^{\simeq}|} \arrow[r] & {|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \to \mathrm{S}_{n-1}\mathcal{D} \times \mathcal{D}))^{\simeq}|} \arrow[r] & {|(\Sigma_\bullet\Sigma_\bullet\mathcal{D})^{\simeq}|}
\end{tikzcd}$$

Hence, to show that the left vertical map is an equivalence, it suffices to argue that the middle vertical is an equivalence. But then we have an equivalence $\Sigma_\bullet(\mathcal{D} \to \mathrm{S}_{n-1}\mathcal{D} \times \mathcal{D})^{\simeq} \simeq \mathrm{S}_{n-1}\mathcal{D}^{\simeq} \times \Sigma_\bullet(\mathcal{D} \xrightarrow{\simeq} \mathcal{D})^{\simeq}$, and so

$$|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \to \mathrm{S}_{n-1}\mathcal{D} \times \mathcal{D}))^{\simeq}| \simeq |(\mathrm{S}_{n-1}\mathcal{D})^{\simeq}| \times |(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \xrightarrow{=} \mathcal{D}))^{\simeq}| \simeq |(\mathrm{S}_{n-1}\mathcal{D})^{\simeq}|.$$

We may now apply Lemma 5.3.14 to obtain the desired equivalence. ☐

To specialise the abstract +=S result to prove Theorem 5.3.17, we will need the following preliminary observation:

Lemma 5.3.16 ([Wal85, Prop. 1.8.9])**.** *Let* $\mathcal{D} \in \mathrm{Wald}$ *admitting pushouts and* $X \in \mathcal{D}$*. The anima* $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ *is contractible if and only if* $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ *is connected and* $|(\Sigma_\bullet\mathcal{D})^{\simeq}| \to |(\Sigma_\bullet\mathcal{D}_X)^{\simeq}|$ *is an equivalence.*

Proof. By Proposition 5.3.9, we have a map of fibre sequences

$$\begin{tikzcd}
{|(\Sigma_\bullet\mathcal{D})^{\simeq}|} \arrow[r] \arrow[d] & {|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \xrightarrow{\simeq} \mathcal{D}))^{\simeq}| \simeq *} \arrow[r] \arrow[d] & {|(\Sigma_\bullet\Sigma_\bullet\mathcal{D})^{\simeq}|} \arrow[d, equal] \\
{|(\Sigma_\bullet\mathcal{D}_X)^{\simeq}|} \arrow[r] & {|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|} \arrow[r] & {|(\Sigma_\bullet\Sigma_\bullet\mathcal{D})^{\simeq}|.}
\end{tikzcd}$$

Thus, if $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ is contractible (in which case it is of course connected), then the middle vertical map is an equivalence, and so the left vertical map is an equivalence. Now for the converse, if $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ is connected, then it is contractible if and only if $|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ is contractible since $\Omega|(\Sigma_\bullet\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}| \simeq \Omega\Sigma|(\Sigma_\bullet(\mathcal{D} \to$

$\mathcal{D}_X))^{\simeq}| \simeq |(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ by group–likeness of $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$. Now since the right horizontal maps in the map of fibre sequences are π_0–surjections (since the targets are connected), we obtain that it is also a bifibre sequence in spectra by Exercise 4.4.2. Hence, since the left vertical map is an equivalence, so is the middle vertical, concluding the proof. ☐

We are finally ready to state and prove the theorem.

Theorem 5.3.17 (Waldhausen's +=S)**.** *Let $\mathcal{D} \in \mathrm{Exact}$ be split. Then the adjunction counit $i_!i^*\mathrm{S}_\bullet\mathcal{D} \to \mathrm{S}_\bullet\mathcal{D}$ induces an equivalence $(\mathcal{D}^{\simeq})^{\mathrm{gp}} \xrightarrow{\simeq} \Omega|(\mathrm{S}_\bullet\mathcal{D})^{\simeq}| \in \mathrm{CGrp}$.*

Proof. We would like to apply Proposition 5.3.15 to show that $|(\Sigma_\bullet\mathcal{D})^{\simeq}| \to |(\mathrm{S}_\bullet\mathcal{D})^{\simeq}|$ is an equivalence, and to do this, we check that the conditions in Lemma 5.3.16 are verified. For each $X \in \mathcal{D}$, the map $|(\Sigma_\bullet\mathcal{D})^{\simeq}| \to |(\Sigma_\bullet\mathcal{D}_X)^{\simeq}|$ is an equivalence by Proposition 5.3.12. To see that $|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$ is connected, recall by construction that $\Sigma_0(\mathcal{D} \to \mathcal{D}_X)^{\simeq} \simeq \mathcal{D}_X^{\simeq}$, $\Sigma_1(\mathcal{D} \to \mathcal{D}_X)^{\simeq} \simeq \mathcal{D}_X^{\simeq} \times \mathcal{D}^{\simeq}$, and that we have identifications

$$d_0 = \mathrm{proj}_1 \colon \mathcal{D}_X^{\simeq} \times \mathcal{D}^{\simeq} \longrightarrow \mathcal{D}_X^{\simeq} \qquad d_1 = \vee \circ (\mathrm{id} \times j) \colon \mathcal{D}_X \times \mathcal{D} \longrightarrow \mathcal{D}_X$$

Hence, by Lemma 5.2.10, we see that

$$\pi_0|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}| \cong \pi_0\mathcal{D}_X / \big[(X \rightarrowtail A) \sim (X \rightarrowtail A \oplus B)\big].$$

Since $\mathcal{D}$ was split, we see that every point $(X \rightarrowtail A)$ in $\pi_0\mathcal{D}_X$ may be written noncanonically as $(X \rightarrowtail X \oplus A/X)$, and so in $\pi_0|(\Sigma_\bullet(\mathcal{D} \to \mathcal{D}_X))^{\simeq}|$, we have $(X \rightarrowtail A) = (X \rightarrowtail X \oplus A/X) \sim (X \xrightarrow{=} X)$, i.e. the anima of interest is connected, as claimed. Applying Ω on both sides concludes the proof. ☐

5.4 Devissage arguments

In this section, we want to prove Quillen's infamous devissage theorem [Qui73]:

Theorem 5.4.1 (Quillen devissage)**.** *Let $\mathcal{A} \subseteq \mathcal{B}$ be an exact inclusion of abelian categories and assume that the following conditions are satisfied:*

1. *$\mathcal{A}$ is closed under subobjects and quotient: if $a \rightarrowtail x \twoheadrightarrow c$ is an exact sequence in $\mathcal{B}$ with $x \in \mathcal{A}$, then both a and c are in $\mathcal{A}$.*
2. *Every object $b \in \mathcal{B}$ admits a finite filtration*

$$0 = b_0 \rightarrowtail b_1 \rightarrowtail \cdots \rightarrowtail b_{n-1} \rightarrowtail b_n = b$$

such that each associated graded b_{i+1}/b_i lies in $\mathcal{A}$.

Then $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ is an equivalence.

This is an example of a result which is most readily proved using the Q-construction rather than the S-construction and we will essentially repeat Quillen's original proof following [Win24, §12] and [Wei13, §V.4]. In fact, to the best of our knowledge a proof using the S-construction does not exist in the literature. Inspired by Theorem 5.2.16 we construct a variant of the Q-construction for exact categories.

Definition 5.4.2. Let $e\colon \Delta \to \Delta$ denote the functor defined by $[n] \mapsto [n]^{\mathrm{op}} * [n] \simeq [2n+1]$. The edgewise subdivision functor is defined by

$$\mathrm{esd} = - \circ e^{\mathrm{op}}\colon \mathrm{sAn} \to \mathrm{sAn}.$$

Lemma 5.4.3. *If $\mathcal{C}$ is an exact category, then $\mathrm{esd}(\mathrm{S}_\bullet\mathcal{C})^{\simeq}$ is a complete Segal anima.*

Proof. See [Win24, Lemma 12.5] for an indication of why this is true. ☐

Definition 5.4.4. Let $\mathrm{Q}\colon \mathrm{Exact} \to \mathrm{Cat}$ denote the functor defined by

$$\mathrm{Exact} \xrightarrow{\mathrm{S}_\bullet(-)^{\simeq}} \mathrm{sAn} \xrightarrow{\mathrm{esd}} \mathrm{sAn} \xrightarrow{\mathrm{ac}} \mathrm{Cat},$$

where $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}$ is the associated category from Construction 4.1.17.

Example 5.4.5. Let $\mathcal{C}$ denote an exact category. An object of $\mathrm{Q}(\mathcal{C})$ is an object of $\mathcal{C}$ and a morphism in $\mathrm{Q}(\mathcal{C})$ between a pair of objects $x, y \in \mathcal{C}$ is a span

$$x \twoheadleftarrow x' \rightarrowtail y$$

where $x \twoheadrightarrow x'$ is a fibration and $x' \rightarrowtail y$ is a cofibration. See the discussion following the proof of [Win24, Lemma 12.5] for a thorough discussion of this.

Observation 5.4.6. Let $\mathcal{A}$ be an abelian category and consider the ∞-category $\mathrm{Q}(\mathcal{A})$ as before. In this case, it turns out that $\mathrm{Q}(\mathcal{A})$ is equivalent to the 1-category whose objects are the objects of $\mathcal{A}$ and morphisms are isomorphism classes of spans $a \twoheadleftarrow b' \rightarrowtail b$. Composition is defined by forming pullbacks of spans and we let $\mathrm{Q}^{\mathrm{cl}}(\mathcal{A})$ denote this 1-category. We again refer to [Win24] for a thorough discussion of this point.

We give the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. By Observation 5.4.6 it suffices to prove that the functor of 1-categories

$$i\colon \mathrm{Q}^{\mathrm{cl}}(\mathcal{A}) \to \mathrm{Q}^{\mathrm{cl}}(\mathcal{B})$$

is an equivalence after realisation. To achieve this it suffices to prove that the comma category i/B is contractible for every $B \in \mathcal{B}$ by Quillen's Theorem A. Recall that i/B is defined by the pullback

$$\begin{tikzcd}
i/B \arrow[r] \arrow[d] & \mathcal{B}_{/B} \arrow[d] \\
\mathrm{Q}^{\mathrm{cl}}(\mathcal{A}) \arrow[r, "i"] & \mathrm{Q}^{\mathrm{cl}}(\mathcal{B})
\end{tikzcd}$$

from which we observe that an object of i/B is a span $a \twoheadleftarrow B' \rightarrowtail B$, where $a \in \mathcal{A}$.

By assumption, we may choose a finite filtration

$$0 = B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{n-1} \rightarrowtail B_n = B$$

such that each B_{i+1}/B_i lies in $\mathcal{A}$. This filtration then induces a sequence of functors

$$i/0 \simeq i/B_0 \to i/B_1 \to \cdots \to i/B_{n-1} \to i/B_n \simeq i/B.$$

where each functor $i/B_i \to i/B_{i+1}$ is given by

$$(a \twoheadleftarrow B' \rightarrowtail B_i) \mapsto (a \twoheadleftarrow B' \rightarrowtail B_i \rightarrowtail B_{i+1}).$$

Note that $|i/0| \simeq *$, so to prove that $|i/B| \simeq *$ we reduce to proving that $|i/B_i| \to |i/B_{i+1}|$ is an equivalence for all i. To ease the notation a bit, we prove that if $b' \rightarrowtail b$ is a monomorphism with $b/b' \in \mathcal{A}$, then $|i/b'| \to |i/b|$ is an equivalence.

Given an object $a \twoheadleftarrow b_2 \rightarrowtail b$ of j/b, write $b_1 = \ker(b_2 \twoheadrightarrow a)$. Let $\mathcal{J}$ denote the full subcategory of i/b spanned by those spans $a \twoheadleftarrow b_2 \rightarrowtail b$ where $b_1 \rightarrowtail b_2 \rightarrowtail b$ factors over $b' \rightarrowtail b$. Since factoring through $b' \rightarrowtail b$ is a property, we obtain a chain of inclusions

$$i/b' \subseteq \mathcal{J} \subseteq i/b.$$

To prove the desired, we construct adjoints of the inclusions. For this, one checks that

$$(i/b) \xrightarrow{\ell} \mathcal{J} \quad :: \quad [a \twoheadleftarrow b_2 \rightarrowtail b] \mapsto [b_2/(b_1 \cap b') \twoheadleftarrow b_2 \rightarrowtail b]$$

$$\mathcal{J} \xrightarrow{r} (i/b') \quad :: \quad [a \twoheadleftarrow b_2 \rightarrowtail b] \mapsto [(b_2 \cap b')/b_1 \twoheadleftarrow b_2 \cap b' \rightarrowtail b']$$

are the left and right adjoints to the respective inclusions. The functor r is well-defined since $(b_2 \cap b')/b_1 \twoheadrightarrow b_2/b_1 \cong a \in \mathcal{A}$ and so is also in $\mathcal{A}$. The functor ℓ is well-defined since $b_2/(b_1 \cap b') \rightarrowtail b_2/b_1 \oplus b/b' \in \mathcal{A}$: this is gotten by considering

$$\begin{tikzcd}
b_1 \cap b' \oplus b' \arrow[r, tail] \arrow[d, tail] & b_2 \oplus b' \arrow[r, two heads] \arrow[d, tail] & b_2/(b_1 \cap b') \arrow[d] \\
b_1 \oplus b' \arrow[r, tail] & b_2 \oplus b \arrow[r, two heads] & b_2/b_1 \oplus b/b'
\end{tikzcd}$$

and applying the snake lemma. □

Our final goal of this section is to discuss the resolution theorem [Qui73, §4, Thm. 3] and explain how to reduce it to the case of filtration length 1 which is proved in [Win24, Theorem 8.11].

Terminology 5.4.7. Let $\mathcal{U} \subseteq \mathcal{C}$ denote a full subcategory of an exact category $\mathcal{C}$. For $n \geq 0$, a $\mathcal{U}$-resolution of length n of an object $x \in \mathcal{C}$ is a sequence of maps

$$0 \rightarrowtail u_n \to u_{n-1} \to \cdots \to u_1 \to u_0 \twoheadrightarrow x$$

in $\mathcal{C}$, where $u_0 \twoheadrightarrow x$ is a cofibration. Note that x has a $\mathcal{U}$-resolution of length 0 precisely if $x \in \mathcal{U}$.

Theorem 5.4.8 (Resolution theorem)**.** *Let $\mathcal{C}$ be an exact category and let $\mathcal{U} \subseteq \mathcal{C}$ be a full subcategory satisfying that:*

(1) $\mathcal{U} \subseteq \mathcal{C}$ is closed under extensions;

(2) for every projection $p\colon u_1 \twoheadrightarrow u_2$ in $\mathcal{C}$ with $u_1, u_2 \in \mathcal{U}$, the fibre of p also lies in $\mathcal{U}$;

(3) for every $x \in \mathcal{C}$, there exists a finite length resolution

$$0 \rightarrowtail u_n \to \cdots \to u_2 \to u_1 \to u_0 \twoheadrightarrow x$$

where $u_i \in \mathcal{U}$ for all i.

Then the inclusion functor induces an equivalence $\mathrm{K}(\mathcal{U}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C})$.

Exercise 5.4.9. The goal of this exercise is to reduce Theorem 5.4.8 to the length 1 case for which the reader can consult [Win24, Theorem 8.11]. We first introduce a bit of notation: For every $n \geq 0$, let $\mathcal{C}_{\mathcal{U}_n}$ denote the full subcategory of $\mathcal{C}$ on those $x \in \mathcal{C}$ which admits a $\mathcal{U}$-resolution of length n.

1. Show that $\mathcal{C}_{\mathcal{U}_0} \simeq \mathcal{U}$.
2. Show that there is a sequence of exact inclusions

$$\mathcal{U} \hookrightarrow \mathcal{C}_{\mathcal{U}_1} \hookrightarrow \mathcal{C}_{\mathcal{U}_2} \hookrightarrow \cdots$$

   of exact categories for every $n \geq 1$.

3. Prove that $\mathcal{C} \simeq \mathrm{colim}(\mathcal{U} \hookrightarrow \mathcal{C}_{\mathcal{U}_1} \hookrightarrow \cdots)$ assuming (3) in Theorem 5.4.8.
4. Prove that $\mathcal{U} \hookrightarrow \mathcal{C}_{\mathcal{U}_1}$ satisfies (2) and (3) above. Deduce that $\mathrm{K}(\mathcal{U}) \to \mathrm{K}(\mathcal{C}_{\mathcal{U}_1})$ is an equivalence by the length 1 case of the resolution theorem and deduce that the same is true for $\mathrm{K}(\mathcal{C}_{\mathcal{U}_{n-1}}) \to \mathrm{K}(\mathcal{C}_{\mathcal{U}_n})$.
5. Use that K-theory preserves filtered colimits to prove Theorem 5.4.8 using the above.

5.5 Interlude: t–structures

Basics

In this section we introduce t-structures on stable ∞-categories, which is a notion that generalises Postnikov towers to an arbitrary stable ∞-categories. We closely follow the canonical source [Lur17, §1.2.1] and [Win24, §10].

Definition 5.5.1. A *t-structure* on a stable ∞-category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ satisfying the following conditions:

(1) If $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, then $\Sigma x \in \mathcal{C}_{\geq 0}$ and $\Sigma^{-1} y \in \mathcal{C}_{\leq 0}$.

(2) If $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, then $\mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq *$.

(3) For every $x \in \mathcal{C}$, there is a fiber sequence

$$x_{\geq 0} \to x \to x_{\leq -1}$$

with $x_{\geq 0} \in \mathcal{C}_{\geq 0}$ and $\Sigma x_{\leq -1} \in \mathcal{C}_{\leq 0}$.

The heart of the t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is defined by $\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$.

Terminology 5.5.2. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure. For every $n \in \mathbb{Z}$, set

$$\begin{aligned} \mathcal{C}_{\geq n} &= \Sigma^n \mathcal{C}_{\geq 0} \\ \mathcal{C}_{\leq n} &= \Sigma^n \mathcal{C}_{\leq 0} \end{aligned}$$

Furthermore, we will use the following terminology:

1. The t-structure is *bounded below* if $\mathcal{C} = \bigcup_{m \in \mathbb{Z}} \mathcal{C}_{\geq m}$.
2. The t-structure is *bounded above* if $\mathcal{C} = \bigcup_{n \in \mathbb{Z}} \mathcal{C}_{\leq n}$.

The t-structure is said to be *bounded* if it is both bounded below and bounded above. In other words, the t-structure is bounded if every object $x \in \mathcal{C}$ lies in $\mathcal{C}_{[m,n]} = \mathcal{C}_{\geq m} \cap \mathcal{C}_{\leq n}$ for a pair of integers $m, n \in \mathbb{Z}$.

Note that condition (2) of Definition 5.5.1 is an assertion about the mapping anima. To the author this is the reasonable condition to impose in the definition of a t-structure but it turns out to be equivalent to a π_0-assertion as we now clarify:

Remark 5.5.3. Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ denote a pair of full subcategories of a stable ∞-category $\mathcal{C}$ and assume that $\mathcal{C}_{\geq 0}$ is closed under Σ as in condition (1) of Definition 5.5.1 above. In this case, condition (2) is equivalent to the seemingly much weaker assertion:

(2′) If $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, then $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq 0$.

We clearly have that (2) ⇒ (2′). Conversely, for every $i \geq 1$, we have that

$$\pi_i \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq \pi_0 \Sigma^{-i} \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1} y) \simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^i x, \Sigma^{-1} y) \simeq 0,$$

where we have used (2′) in the last equivalence since $\Sigma^i x \in \mathcal{C}_{\geq 0}$ by assumption. Note that this justifies that Definition 5.5.1 is equivalent to the definition given in [Lur17, Definition 1.2.1.4] namely that a t-structure on a stable ∞-category is the datum of a t-structure on its homotopy category considered as a triangulated category. We prefer Definition 5.5.1 to avoid any mention of homotopy categories.

Example 5.5.4. The datum of a t-structure on $\mathcal{C}$ enforces strong rigidity results for $\mathcal{C}$: namely, the heart of the t-structure $\mathcal{C}^\heartsuit$ is a 1-category. For $x, y \in \mathcal{C}^\heartsuit$, we have that

$$\pi_i \operatorname{Map}_{\mathcal{C}^\heartsuit}(x,y) \simeq \pi_i \operatorname{Map}_{\mathcal{C}}(x,y) \simeq 0$$

for $i \geq 1$ by condition (2) of Definition 5.5.1 since $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$. It follows that $\mathcal{C}^\heartsuit$ is a 1-category with hom-set given by

$$\operatorname{Hom}_{\mathcal{C}^\heartsuit}(x,y) = \pi_0 \operatorname{Map}_{\mathcal{C}}(x,y).$$

We will later see that $\mathcal{C}^\heartsuit$ is not just a 1-category but an example of an abelian category. See also Exercise 5.6.3 for another example of a rigidity result enforced by the datum of a *bounded* t-structure.

Observation 5.5.5. Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Recall that the mapping anima functor canonically refines to a mapping spectrum functor

$$\operatorname{map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$$

with $\operatorname{Map}_{\mathcal{C}} \simeq \Omega^\infty \operatorname{map}_{\mathcal{C}}$. Condition (2) in Definition 5.5.1 is equivalent to the assertion:

(2") If $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$, then $\operatorname{map}_{\mathcal{C}}(x,y)$ is coconnective.

It can sometimes be useful to phrase (2) in this fashion. This also further cements that t-structures and weight structures are dual since for weight structures we require that this mapping spectrum is connective.

Before proceeding we record some examples and we recommend that the reader carefully checks that the conditions of Definition 5.5.1 hold.

Exercise 5.5.6 (Self–duality of t–structures)**.** If $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is a t-structure on a stable ∞-category $\mathcal{C}$, show that $\mathcal{C}^{\mathrm{op}}$ admits a t-structure with $(\mathcal{C}^{\mathrm{op}})_{\geq 0} = \mathcal{C}^{\mathrm{op}}_{\leq 0}$ and $(\mathcal{C}^{\mathrm{op}})_{\leq 0} = \mathcal{C}^{\mathrm{op}}_{\geq 0}$. Note that the t-structure on $\mathcal{C}^{\mathrm{op}}$ is bounded above/below precisely if the t-structure on $\mathcal{C}$ is bounded below/above.

Example 5.5.7. The stable ∞-category Sp of spectra admits a t-structure $(\mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0})$, where $\mathrm{Sp}_{\geq 0}$ denotes the full subcategory of connective spectra while $\mathrm{Sp}_{\leq 0}$ denotes the full subcategory of coconnective spectra. Note that this t-structure is not bounded. We have that $\mathrm{Sp}^\heartsuit \simeq \mathrm{Ab}$ since the 1-category of abelian groups is equivalent to the full subcategory of spectra spanned by those spectra which only have a single homotopy group in degree 0.

Example 5.5.8. Let R be a commutative ring. The derived ∞-category $\mathcal{D}(R)$ admits a t-structure, where $\mathcal{D}(R)_{\geq 0}$ is the full subcategory spanned by those $M \in \mathcal{D}(R)$ with $H_i(M) = 0$ for $i < 0$ and $\mathcal{D}(R)_{\leq 0}$ is the full subcategory spanned by those $M \in \mathcal{D}(R)$ with $H_i(M) = 0$ for $i > 0$. Every $M \in \mathcal{D}(R)$ can be represented by a complex

$$\cdots \to M_2 \to M_1 \xrightarrow{d_0} M_0 \xrightarrow{d_{-1}} M_{-1} \to \cdots$$

In this case, $M_{\geq 0}$ is represented by the complex

$$\cdots \to M_1 \xrightarrow{d_0} \ker(d_{-1}) \to 0 \to 0 \to \cdots$$

while $M_{\leq -1}$ is represented by the complex

$$\cdots \to 0 \to 0 \to M_0/\mathrm{im}(d_0) \to M_{-1} \to \cdots .$$

The heart of this t-structure is equivalent to the ordinary category of R-modules.

The following result shows that t-structures are overdetermined in the sense that one of the full subcategories of a t-structure determines the other one.

Lemma 5.5.9. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then*

(1) $\mathcal{C}_{\geq 0} = \{x \in \mathcal{C} \mid \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1}y) \simeq * \text{ for all } y \in \mathcal{C}_{\leq 0}\}$,

(2) $\mathcal{C}_{\leq 0} = \{y \in \mathcal{C} \mid \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1}y) \simeq * \text{ for all } x \in \mathcal{C}_{\geq 0}\}$.

Proof. We prove (1) since (2) follows by duality. First note that $\mathcal{C}_{\geq 0}$ is a subset of the right hand side by definition. Therefore, let $x \in \mathcal{C}$ and assume that $\mathrm{Map}_{\mathcal{C}}(x, y) \simeq *$ for every $y \in \mathcal{C}_{\leq 0}$. Again by definition, there is a cofiber sequence

$$x_{\geq 0} \to x \to x_{\leq -1},$$

where $x_{\geq 0} \in \mathcal{C}_{\geq 0}$ and $x_{\leq -1} \in \mathcal{C}_{\leq -1}$. The functor $\mathrm{Map}_{\mathcal{C}}(-, x_{\leq -1})$ carries cofiber sequences to fiber sequences, so we obtain a fiber sequence of anima

$$\mathrm{Map}_{\mathcal{C}}(x_{\leq -1}, x_{\leq -1}) \to \mathrm{Map}_{\mathcal{C}}(x, x_{\leq -1}) \to \mathrm{Map}_{\mathcal{C}}(x_{\geq 0}, x_{\leq -1}),$$

where the middle and right terms are both contractible by definition. It follows that $\mathrm{Map}_{\mathcal{C}}(x_{\leq -1}, x_{\leq -1}) \simeq *$ which in particular means that $\mathrm{id}_{x_{\leq -1}}$ is nullhomotopic which yields that $x_{\leq -1} \simeq 0$. From the first fiber sequence, we conclude that $x_{\geq 0} \to x$ is an equivalence which proves that $x \in \mathcal{C}_{\geq 0}$ as desired. □

Remark 5.5.10. By Remark 5.5.3 and Lemma 5.5.9, we equivalently obtain that

(1) $\mathcal{C}_{\geq 0} = \{x \in \mathcal{C} \mid \pi_0 \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1}y) \simeq 0 \text{ for all } y \in \mathcal{C}_{\leq 0}\}$,

(2) $\mathcal{C}_{\leq 0} = \{y \in \mathcal{C} \mid \pi_0 \mathrm{Map}_{\mathcal{C}}(x, \Sigma^{-1}y) \simeq 0 \text{ for all } x \in \mathcal{C}_{\geq 0}\}$.

which is often also useful.

Proposition 5.5.11. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.*

1. *There is a pair of adjunctions*

$$\begin{tikzcd} \mathcal{C}_{\geq m} \arrow[r, hook, shift left] & \mathcal{C} \arrow[l, shift left, "\tau_{\geq m}"] \arrow[r, shift left, "\tau_{\leq n}"] & \mathcal{C}_{\leq n} \arrow[l, hook, shift left] \end{tikzcd}$$

where $\tau_{\geq m}$ is right adjoint of $\mathcal{C}_{\geq m} \subseteq \mathcal{C}$ and $\tau_{\leq n}$ is left adjoint of $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$.

2. *The canonical transformation*

$$\tau_{\leq n} \circ \tau_{\geq m} \to \tau_{\geq m} \circ \tau_{\leq n} \colon \mathcal{C} \to \mathcal{C}_{\geq m} \cap \mathcal{C}_{\leq n}$$

is an equivalence for all m and n.

Proof. We prove (1) and refer to [Lur17, Proposition 1.2.1.10] or [Win24, Proposition 10.5] for a proof of the second assertion. It suffices to construct the adjoints for $m = n = 0$ since the adjoints for arbitrary m and n are obtained by taking (de)suspensions.

Let $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}$. By definition, there is a fiber sequence

$$y_{\geq 0} \to y \to y_{\leq -1}$$

with $y \in \mathcal{C}_{\geq 0}$ and $y_{\leq -1} \in \mathcal{C}_{\leq -1}$. Applying $\mathrm{Map}_{\mathcal{C}}(x, -)$ we obtain a fiber sequence

$$\mathrm{Map}_{\mathcal{C}}(x, y_{\geq 0}) \to \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, y_{\leq -1}),$$

where the right term is contractible by definition. It follows that $y_{\geq 0}$ is a right adjoint object to y. □

By Definition 5.5.1, for every $x \in \mathcal{C}$, there is a fiber sequence

$$x_{\geq 0} \to x \to x_{\leq -1}$$

with $x_{\geq 0} \in \mathcal{C}_{\geq 0}$ and $\Sigma x_{\leq -1} \in \mathcal{C}_{\leq 0}$. The content of Proposition 5.5.11 is that this fiber sequence is functorial in x and is equivalent to the following fiber sequence

$$\tau_{\geq 0} x \to x \to \tau_{\leq -1} x.$$

Let us warn the reader that this is yet another subtle difference between t-structures and weight structures in which the weight truncations are not functorial.

Definition 5.5.12. Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Define a functor $\pi_0 \colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ by the following formula

$$\pi_0 = \tau_{\geq 0} \circ \tau_{\leq 0} \simeq \tau_{\leq 0} \circ \tau_{\geq 0} \colon \mathcal{C} \to \mathcal{C}^{\heartsuit}.$$

For every $n \in \mathbb{Z}$, define $\pi_n \colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ by $\pi_n = \pi_0 \circ \Sigma^{-n}$.

Next, we analyse the heart of a t-structure.

Proposition 5.5.13 ([Win24, Cor. 10.6])**.** *Let $\mathcal{C}$ be stable with a t–structure. Then $\mathcal{C}^{\heartsuit}$ is an abelian category.*

Proof. Since both $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ are closed under direct sums, $\mathcal{C}^{\heartsuit}$ is an additive category. And by Example 5.5.4 we even know that it is a 1–category. The composition of inclusions $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ consists of a Bousfield colocalisation and a Bousfield localisation, so $\mathcal{C}^{\heartsuit}$ is finitely complete and cocomplete. Concretely, the kernel and cokernel are computed as follows: if $i\colon a \to b$ and $p\colon b \to c$ are maps in $\mathcal{C}^{\heartsuit}$, then

$$\begin{aligned}\ker(p\colon b \to c) &\simeq \tau_{\geq 0}\operatorname{fib}(p\colon b \to c)\\ \operatorname{coker}(i\colon a \to b) &\simeq \tau_{\leq 0}\operatorname{cofib}(i\colon a \to b),\end{aligned}$$

where the fibre and cofibre are calculated in $\mathcal{C}$. To show that it is an abelian category, we are left to showing that every monomorphism (resp. epimorphism) is a kernel (resp. cokernel) of some morphism.

We only show the case of monomorphisms since the other case is dual. Let $i\colon a \to b$ be a monomorphism in $\mathcal{C}^{\heartsuit}$ and denote by $p\colon b \to c$ the cofibre of i in $\mathcal{C}$. Then $c \in \mathcal{C}_{\geq 0}$. Now, for $x \in \mathcal{C}_{\geq 0}$ we have an exact sequence

$$0 = \pi_0 \operatorname{Map}_{\mathcal{C}}(x, \Sigma^{-1}b) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(x, \Sigma^{-1}c) \to \pi_0 \operatorname{Map}_{\mathcal{C}}(x, a) \xrightarrow{i_*} \pi_0 \operatorname{Map}_{\mathcal{C}}(x, b)$$

Since i was a monomorphism, i_* is injective and so the left map is a surjection from 0. Hence, $\pi_0 \operatorname{Map}_{\mathcal{C}}(x, \Sigma^{-1}c) = 0$, i.e. c is also 0–truncated. Therefore, by the formula for the cokernels above, we see that $c \simeq \operatorname{coker}(i\colon a \to b)$. In particular, we see that

$$a \simeq \operatorname{fib}(b \to c) \simeq \tau_{\geq 0}\operatorname{fib}(b \to c) \simeq \ker(b \to c)$$

as required. In addition, note also that $b \to c$ is an epimorphism in $\mathcal{C}^{\heartsuit}$ since for any $y \in \mathcal{C}^{\heartsuit}$, we have the fibre sequence

$$0 = \operatorname{Map}_{\mathcal{C}^{\heartsuit}}(\Sigma a, y) \longrightarrow \operatorname{Map}_{\mathcal{C}^{\heartsuit}}(c, y) \longrightarrow \operatorname{Map}_{\mathcal{C}^{\heartsuit}}(b, y)$$

and so $\operatorname{Map}_{\mathcal{C}^{\heartsuit}}(c, y) \to \operatorname{Map}_{\mathcal{C}^{\heartsuit}}(b, y)$ is an injection. □

Proposition 5.5.14. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Every fiber sequence $x \to y \to z$ in $\mathcal{C}$ induces a long exact sequence*

$$\cdots \to \pi_{n+1}x \to \pi_{n+1}y \to \pi_{n+1}z \to \pi_n x \to \cdots$$

in the abelian category $\mathcal{C}^{\heartsuit}$.

Proof. See [Win24, Proposition 10.8]. □

To prepare for the proof of the theorem of the heart later, we will also need the following observation.

Proposition 5.5.15. *Suppose a small stable category $\mathcal{C}$ is equipped with a bounded below t–structure. Then the inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ induces an equivalence*

$$\operatorname{colim}\left(\mathcal{C}_{\geq 0} \overset{\Sigma}{\hookrightarrow} \mathcal{C}_{\geq 0} \overset{\Sigma}{\hookrightarrow} \mathcal{C}_{\geq 0} \overset{\Sigma}{\hookrightarrow}\right) \xrightarrow{\simeq} \mathcal{C}$$

Proof. First, note that since $\mathcal{C}$ was stable, the suspension maps $\Sigma\colon \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$ are fully faithful, and so the colimit is computed by taking the union along the fully faithful inclusions. On the other hand, the comparison map from the colimit to $\mathcal{C}$ is induced by the diagram (which is already a coherent diagram in the ∞–categorial sense by Observation 3.2.1)

$$\begin{tikzcd}
\mathcal{C}_{\geq 0} \arrow[r, hook, "\Sigma"] \arrow[d, hook] & \mathcal{C}_{\geq 0} \arrow[r, hook, "\Sigma"] \arrow[d, hook, "\Omega"] & \mathcal{C}_{\geq 0} \arrow[r, hook, "\Sigma"] \arrow[d, hook, "\Omega^2"] & \cdots \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \cdots
\end{tikzcd}$$

This comparison map is automatically also fully faithful since the colimit was computed by taking unions. To see that it is essentially surjective, just note that every object in $\mathcal{C}$ is in the image of $\Omega^n\colon \mathcal{C}_{\geq 0} \to \mathcal{C}$ for some n since the t–structure was bounded below. This concludes the proof. □

To end this part, we say a bit about t–structures on large categories. The following pair of results give us a huge source of t–structures for presentable categories, but we will neither prove them in this course nor state them in their strongest form. We refer the reader to the cited references for more details.

Definition 5.5.16. Let $\mathcal{C}, \mathcal{D}$ be stable categories equipped with t–structures. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is *right t–exact (resp. left t–exact)* if $F(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq 0}$ (resp. $F(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq 0}$. We say that it is *$t$–exact* if it is both right and left t–exact.

Terminology 5.5.17. A t–structure on a presentable stable category $\mathcal{E}$ is said to be *accessible* if $\mathcal{E}_{\geq 0}$ is also presentable.

Proposition 5.5.18 (t–structure from presentable subcategories, [Lur17, Prop. 1.4.4.11])**.** *Let $\mathcal{C}$ be a presentable stable category. If $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory which is presentable, closed under colimits, and closed under extensions, then there exists an accessible t–structure on $\mathcal{C}$ such that $\mathcal{C}_{\geq 0} = \mathcal{C}'$.*

Proposition 5.5.19 (t–structure from Ind–completions, [Lur18b, Lem. C.2.4.3], [AGH18, Prop. 2.13])**.** *Suppose that $\mathcal{C}$ is a small stable category with a t–structure. Then $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \subseteq \mathrm{Ind}(\mathcal{C})$ determines the nonnegative part of an accessible t–structure such that the inclusion $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ is t–exact.*

Interactions with Waldhausen structures

Exercise 5.5.20. Let $\mathcal{C} \in \mathrm{Exact}$ and $\mathcal{U} \subseteq \mathcal{C}$ an additive full subcategory closed under extensions, i.e. for every exact sequence $u \rightarrowtail y \twoheadrightarrow w$ with $u, w \in \mathcal{U}$, the object y is also in $\mathcal{U}$. Then show that $\mathcal{U}$ inherits an exact structure from $\mathcal{C}$ by setting

$$\mathrm{co}\mathcal{U} := \{i \in \mathrm{co}\mathcal{C} \cap \mathcal{U} \mid \mathrm{cofib}(i) \in \mathcal{U}\} \qquad \mathrm{pr}\mathcal{U} := \{p \in \mathrm{pr}\mathcal{C} \cap \mathcal{U} \mid \mathrm{fib}(p) \in \mathcal{U}\}.$$

Furthermore, under these structures, the inclusion $\mathcal{U} \subseteq \mathcal{C}$ is an exact functor.

Proposition 5.5.21 (Exact structures from t–structures)**.** *Let $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t–structure on a stable category $\mathcal{C}$. Then $(\mathcal{C}_{\leq 0}, \mathrm{co}\mathcal{C}_{\leq 0}, \mathcal{C}_{\leq 0})$ and $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\geq 0}, \mathrm{pr}\mathcal{C}_{\geq 0})$ are exact structures which make the inclusions $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ and $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ into exact functors. Furthermore, $\mathrm{co}\mathcal{C}_{\leq 0}$ may alternatively be described as those morphisms which are fibres of morphisms in $\mathcal{C}_{\leq 0}$, and similarly, $\mathrm{pr}\mathcal{C}_{\leq 0}$ may be described as those morphisms which are cofibres of morphisms in $\mathcal{C}_{\geq 0}$.*

Proof. We only do the cofibration case since the other is similar. Since $\mathcal{C}$ was stable, it has the canonical exact structure $(\mathcal{C}, \mathcal{C}, \mathcal{C})$, and so by Exercise 5.5.20, we only have to show that:

(a) the inclusion $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under extensions;

(b) $\mathrm{co}\mathcal{C}_{\leq 0}$, as defined in Exercise 5.5.20, consists precisely of those morphisms which are fibres of morphisms in $\mathcal{C}_{\leq 0}$;

(c) all morphisms in $\mathcal{C}_{\leq 0}$ have fibres (computed in $\mathcal{C}$) in $\mathcal{C}_{\leq 0}$.

Points (a) and (c) are clear from the long exact sequence of homotopy groups from Proposition 5.5.14. For (b), by (c) we know that if $f\colon x \to y$ is a morphism in $\mathcal{C}_{\leq 0}$, then $\mathrm{fib}(f) \to x$ will still be in $\mathcal{C}_{\leq 0}$. Conversely, if $i\colon u \to x$ is a morphism in $\mathcal{C}_{\leq 0}$ such that $x \to \mathrm{cofib}(i)$ is again in $\mathcal{C}_{\leq 0}$, then of course $u \simeq \mathrm{fib}(x \to \mathrm{cofib}(i))$. This completes the proof. □

Remark 5.5.22. Note that under these exact structures, the suspension functor $\Sigma\colon \mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$ is an exact functor.

Exercise 5.5.23. Show using Exercise 5.5.20 that $\mathcal{C}^{\heartsuit} \subseteq \mathcal{C}_{\leq 0}$ inherits the exact structure on $\mathcal{C}_{\leq 0}$ from Proposition 5.5.21.

5.6 The theorem of the t–heart

Using the exact structures on the inclusions $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$, $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$, and $\mathcal{C}^{\heartsuit} \subseteq \mathcal{C}_{\leq 0}$ from Proposition 5.5.21 and Exercise 5.5.23, we maps on the K–theory spectra

$$\mathrm{K}(\mathcal{C}_{\geq 0}) \to \mathrm{K}(\mathcal{C}) \qquad \mathrm{K}(\mathcal{C}_{\leq 0}) \to \mathrm{K}(\mathcal{C}) \qquad \mathrm{K}(\mathcal{C}^{\heartsuit}) \to \mathrm{K}(\mathcal{C}).$$

The business of this section is to show that these maps are equivalences under boundedness conditions on the t–structures. To motivate the intuition, we first give the π_0–statement, which should already provide the moral reason that the theorem of the t–heart could be true. We learnt it from [AGH18, Lem. 2.11].

Proposition 5.6.1. *Let $\mathcal{E}$ be a small stable category equipped with a bounded t–structure. Then the natural map $\mathrm{K}_0(\mathcal{E}^{\heartsuit}) \to \mathrm{K}_0(\mathcal{E})$ is an isomorphism.*

Proof. By functoriality of the t–structure fibre sequences, we may check that the map

$$\mathrm{K}_0(\mathcal{E}) \longrightarrow \mathrm{K}_0(\mathcal{E}^\heartsuit) \quad :: \quad [x] \mapsto \sum_{n\in\mathbb{Z}} (-1)^n [\pi_n x]$$

is a well–defined homomorphism. It is easy to see that both compositions give the identity maps. □

We illustrate how one can hope to use theorems of the t–heart in proving something which has nothing to do with K–theory. The following funny proposition was first observed by Antieau–Gepner–Heller (cf. [AGH18, Cor. 2.14]). First, we will need an exercise:

Exercise 5.6.2. Show that abelian categories are idempotent–complete. **Hint:** *for an idempotent $e\colon A \to A$ in an abelian category, use that it factors as a composition of $\pi\colon A \twoheadrightarrow \operatorname{coker}(\ker e)$ with $i\colon \operatorname{coker}(\ker e) \rightarrowtail A$. Using that $\twoheadrightarrow$ and $\rightarrowtail$ are epimorphisms and monomorphisms, respectively, deduce from $e^2 = e$ that $\pi \circ i = \mathrm{id}$.*

Proposition 5.6.3. *Let $\mathcal{C}$ be a stable ∞-category with a bounded t-structure. Then $\mathcal{C}$ is idempotent–complete.*

Proof. Let $\mathcal{D} \coloneqq \mathcal{C}^\natural \simeq \mathrm{Ind}(\mathcal{C})^\omega$. We first claim that the bounded t–structure on $\mathcal{C}$ extends to one on $\mathcal{D}$. We have to show that the functors $\tau_{\leq 0}$ and $\tau_{\geq 0}$ on $\mathrm{Ind}(\mathcal{C})$ coming from the induced t–structure on $\mathrm{Ind}(\mathcal{C})$ from Proposition 5.5.19 preserve compact objects. But this is clear since $\tau_{\leq 0}$ and $\tau_{\geq 0}$ are functors, and so they preserve retracts. Hence, the induced t–structure on $\mathrm{Ind}(\mathcal{C})$ restricts to one on $\mathcal{D}$.

Next, it is not hard to check that $\mathcal{D}^\heartsuit$ is the idempotent–completion of $\mathcal{C}^\heartsuit$. But then since abelian categories are idempotent–complete by Exercise 5.6.2, we see that $\mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is an equivalence. Therefore, by Proposition 5.6.1, we get that $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}(\mathcal{D})$ is an isomorphism. By Theorem 4.5.2, we must have that $\mathcal{C} \to \mathcal{D}$ was an equivalence. □

Remark 5.6.4. In fact, one does not need K–theory to prove the result above, and the point of the proof was just to illustrate that it can be convenient to know some K–theory (and maybe even better, that knowing some K–theory can suggest that certain things should be true). In [AGH18, Cor. 2.14], they also included a proof without resorting to Theorem 4.5.2, which we can recommend as something which is also instructive to see.

Proposition 5.6.5 ([Win24, Cor. 3.12])**.** *Let $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t–structure on a stable category $\mathcal{C}$ and $H\colon \mathrm{Wald} \to \mathrm{An}$ be an additive invariant. Then:*

1. *The suspension functor induces an equivalence $H(\Sigma)\colon H(\mathcal{C}_{\geq 0}) \xrightarrow{\simeq} H(\mathcal{C}_{\geq 0})$,*
2. *If the t–structure is bounded below, then the canonical map $H(\mathcal{C}_{\geq 0}) \to H(\mathcal{C})$ is an equivalence.*

Proof. For part (1), by similar methods as in Corollary 4.3.3 using instead Theorem 5.2.15, we see that the cofibre sequences of exact functors $\mathcal{C}_{\geq 0} \to \mathcal{C}_{\geq 0}$

$$\mathrm{id} \Rightarrow 0 \Rightarrow \Sigma \qquad \Sigma \Rightarrow 0 \Rightarrow \Sigma^2$$

implies that $H(\Sigma) + H(\mathrm{id}) \simeq 0 \simeq H(\Sigma) + H(\Sigma^2)$. Hence in total we see that

$$\mathrm{id}_{H(\mathcal{C}_{\geq 0})} \simeq H(\mathrm{id}) \simeq H(\mathrm{id}) + H(\Sigma) + H(\Sigma^2) \simeq \mathrm{K}(\Sigma^2)$$

Thus $H(\Sigma)\colon H(\mathcal{C}_{\geq 0}) \to H(\mathcal{C}_{\geq 0})$ is an equivalence as desired. Part (2) is then an immediate combination of (1), that $H(-)$ preserves filtered colimits, and that $\mathcal{C} \simeq \mathrm{colim}(\mathcal{C}_{\geq 0} \xrightarrow{\Sigma} \mathcal{C}_{\geq 0} \xrightarrow{\Sigma} \mathcal{C}_{\geq 0} \cdots)$ from Proposition 5.5.15 by the bounded below hypothesis. ☐

Construction 5.6.6 (Relative S–construction, [Win24, Def. 7.5])**.** Let $\mathcal{C} \in \mathrm{Wald}$ and $\mathcal{U} \subseteq \mathcal{C}$ a full Waldhausen subcategory. Define the *relative S–construction* of $\mathcal{U} \subseteq \mathcal{C}$ as the pullback

$$\begin{tikzcd}
\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{D\acute{e}cS}_\bullet\mathcal{C} \arrow[d, "d_\bullet"] \\
\mathrm{S}_\bullet\mathcal{U} \arrow[r] & \mathrm{S}_\bullet\mathcal{C}.
\end{tikzcd} \tag{5.5}$$

Be warned that here we have used the map $d_\bullet$ (i.e. d_{n+1} at level n) for the right vertical map instead of the usual d_0 as in the relative Q–construction from the proof of Theorem 4.4.6 or the relative suspension Construction 5.3.6! This is geared to the requirements of the proof of the theorem of the heart.

The transformation $\mathrm{const}_\bullet\,\mathcal{C} \to \mathrm{D\acute{e}cS}_\bullet\mathcal{C}$ induced by the identity map $\mathcal{C} \xrightarrow{=} \mathrm{S}_1\mathcal{C}$ induces a natural transformation

$$\mathrm{const}_\bullet\,\mathcal{C} \longrightarrow \mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C}).$$

To unwind the relative S–construction, note that since the bottom horizontal map $\mathrm{S}_\bullet\mathcal{U} \to \mathrm{S}_\bullet\mathcal{C}$ is pointwise fully faithful, so must the map $\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C}) \to \mathrm{D\acute{e}cS}_\bullet\mathcal{C}$. That is, $\mathrm{S}_n(\mathcal{U} \subseteq \mathcal{C})$ is a full subcategory of $\mathrm{S}_{n+1}\mathcal{C}$. Since the right vertical map d_n in (5.5) is given by forgetting the rightmost column, an object in $\mathrm{S}_n(\mathcal{U} \subseteq \mathcal{C})$ thus looks like

$$\begin{tikzcd}
u_{0,1} \arrow[r, tail] & u_{0,2} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & u_{0,n} \arrow[r, tail] \arrow[d, two heads] & x_{0,n+1} \arrow[d, two heads] \\
& u_{1,2} \arrow[r, tail] & \cdots \arrow[r, tail] & u_{1,n} \arrow[r, tail] \arrow[d, two heads] & x_{1,n+1} \arrow[d, two heads] \\
& & & \vdots & \vdots \arrow[d, two heads] \\
& & & & x_{n,n+1}
\end{tikzcd}$$

where the $u_{i,j}$'s are in $\mathcal{U}$.

Observation 5.6.7. For $n, k \in \mathbb{N}$, we claim that

$$\mathrm{S}_n\mathrm{S}_k(\mathcal{U} \subseteq \mathcal{C}) \simeq \mathrm{S}_k(\mathrm{S}_n\mathcal{U} \subseteq \mathrm{S}_n\mathcal{C}).$$

To see this, viewing $\mathrm{S}_n\mathrm{S}_k(\mathcal{U} \subseteq \mathcal{C}) \subseteq \mathrm{Fun}([n-1] \times [k], \mathcal{C})$ and the n and k variables in the horizontal and vertical directions respectively, an object in $\mathrm{S}_n\mathrm{S}_k(\mathcal{U} \subseteq \mathcal{C})$ looks like

$$\begin{tikzcd}
x_{0,0} \arrow[r, two heads] \arrow[d, two heads] & x_{1,0} \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & x_{n-1,0} \arrow[d, two heads] \\
x_{0,1} \arrow[r, two heads] \arrow[d, two heads] & x_{1,1} \arrow[r, two heads] \arrow[d, two heads] & \cdots \arrow[r, two heads] \arrow[d, two heads] & x_{n-1,1} \arrow[d, two heads] \\
\vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] & \vdots & \vdots \arrow[d, two heads] \\
x_{0,k} \arrow[r, two heads] & x_{1,k} \arrow[r, two heads] & \cdots \arrow[r, two heads] & x_{n-1,k}
\end{tikzcd}$$

where the vertical fibres are in $\mathcal{U}$ and where the horizontal maps of the columns are fibrations in $\mathrm{S}_{k+1}\mathcal{C}$. By definition of the Waldhausen structures on the S–construction from the dual of Construction 5.2.9 (using projections instead of cofibrations), this latter condition is equivalent to the vertical maps of rows being fibrations in $\mathrm{S}_n\mathcal{C}$. Hence, we obtain the identification $\mathrm{S}_n\mathrm{S}_k(\mathcal{U} \subseteq \mathcal{C}) \simeq \mathrm{S}_k(\mathrm{S}_n\mathcal{U} \subseteq \mathrm{S}_n\mathcal{C})$ as desired.

Theorem 5.6.8 (Generic fibration theorem for Waldhausen categories, [Win24, Thm. 7.7])**.** *Let $\mathcal{C} \in \mathrm{Wald}$ and $\mathcal{U} \subseteq \mathcal{C}$ be a full Waldhausen subcategory. Let $G\colon \mathrm{Wald} \to \mathrm{An}$ be an additive invariant (c.f. Definition 5.2.13). Then we have a fibre sequence*

$$G\mathcal{U} \longrightarrow G\mathcal{C} \longrightarrow |G\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C})|$$

in CGrp.

Idea 5.6.9 (The generic fibration strategy, [Win24, Rmk. 7.9])**.** We extract a concrete strategy as to how we could use the generic fibration Theorem 5.6.8 for the case when the additive invariant is $G = |(\mathrm{S}_\bullet -)^{\simeq}|$. The input to the strategy will be the following:

(1) Assume that $\mathcal{U} \subseteq \mathcal{C}$ is closed under extensions. Then it is a good exercise to see that the collection of morphisms

$$\mathrm{pr}_{\mathcal{U}}\mathcal{C} \coloneqq \{p\colon x \twoheadrightarrow y \mid p \in \mathrm{pr}\mathcal{C} \text{ and } \mathrm{fib}(p) \in \mathcal{U}\}$$

forms a subcategory of $\mathcal{C}$ (i.e. $\mathrm{pr}_{\mathcal{U}}\mathcal{C}$ is closed under compositions).

(2) We have an essentially surjective functor $P\colon \mathcal{C} \to \mathcal{D}$ which inverts morphisms in $\mathrm{pr}_{\mathcal{U}}\mathcal{C}$.

The strategy is a bit long to explain, but the endpoint to keep in mind is that the upshot will be that we just have to show that a canonically constructed map (5.7) is an equivalence.

Now to begin explaining the strategy, write $\mathrm{Fun}^{\simeq} \subseteq \mathrm{Fun}$ for the full subcategory of functors which inverts all morphisms. Then by our concrete description of $\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C})$ from Construction 5.6.6, the functor P induces a functor

$$P\colon \mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C}) \longrightarrow \mathrm{Fun}^{\simeq}([\bullet], \mathcal{D}) \simeq \mathrm{Fun}(|[\bullet]|, \mathcal{D}) \simeq \mathrm{const}_\bullet \mathcal{D}$$

where the equivalence is since $|[n]| \simeq *$ for all n. Applying $(\mathrm{S}_\bullet -)^{\simeq}$ to this map yields

$$\mathrm{S}_\bullet \mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C})^{\simeq} \longrightarrow (\mathrm{S}_\bullet \, \mathrm{const}_\bullet \mathcal{D})^{\simeq}. \tag{5.6}$$

This map (5.6) will be our candidate to describe the generic cofibre from Theorem 5.6.8 as the K–theory of another Waldhausen category. If we could show that this is an equivalence upon applying $|-|$, then by Theorem 5.6.8, we will have produced a fibre sequence

$$\mathrm{K}(\mathcal{U}) \longrightarrow \mathrm{K}(\mathcal{C}) \xrightarrow{P} \mathrm{K}(\mathcal{D})$$

in CGrp. Furthermore, since $P\colon \mathcal{C} \to \mathcal{D}$ was essentially surjective, the map $\mathrm{K}_0(\mathcal{C}) \to \mathrm{K}_0(\mathcal{D})$ will be surjective, and so by Exercise 4.4.7, this is even a fibre sequence in Sp.

To make this work, fix an n. By our concrete description of $\mathrm{S}_\bullet(\mathrm{S}_n\mathcal{U} \subseteq \mathrm{S}_n\mathcal{C})$, we have

$$\mathrm{S}_\bullet(\mathrm{S}_n\mathcal{U} \subseteq \mathrm{S}_n\mathcal{C})^{\simeq} \simeq \mathrm{N}(\mathrm{pr}_{\mathrm{S}_n\mathcal{U}}\mathrm{S}_n\mathcal{C})$$

where $\mathrm{N}(\mathrm{pr}_{\mathrm{S}_n\mathcal{U}}\mathrm{S}_n\mathcal{C}) \in \mathrm{sAn}$ is the Rezk nerve of the nonfull subcategory $\mathrm{pr}_{\mathrm{S}_n\mathcal{U}}\mathrm{S}_n\mathcal{C} \subset \mathrm{S}_n\mathcal{C}$. Therefore, applying $(-)^{\simeq}$ to (5.6) and using Observation 5.6.7 that $\mathrm{S}_n\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C})^{\simeq} \simeq \mathrm{S}_\bullet(\mathrm{S}_n\mathcal{U} \subseteq \mathrm{S}_n\mathcal{C})^{\simeq}$, we get the map $\mathrm{N}(\mathrm{pr}_{\mathrm{S}_n\mathcal{U}}\mathrm{S}_n\mathcal{C}) \longrightarrow \mathrm{N}(\mathrm{S}_n\mathcal{D}^{\simeq})$. Upon applying $|\mathrm{ac}(-)|$, i.e. geometrically realising the associated category, we finally obtain

$$|\mathrm{pr}_{\mathrm{S}_n\mathcal{U}}\mathrm{S}_n\mathcal{C}| \longrightarrow (\mathrm{S}_n\mathcal{D})^{\simeq}. \tag{5.7}$$

If we can show that (5.7) is an equivalence for all n, then we will have shown that for all n, $|\mathrm{S}_n\mathrm{S}_\bullet(\mathcal{U} \subseteq \mathcal{C})^{\simeq}| \longrightarrow |\mathrm{S}_n(\mathrm{const}_\bullet \mathcal{D})^{\simeq}|$ is an equivalence, and hence that (5.6) is an equivalence upon applying $|-|$, as desired. This concludes the strategy.

Theorem 5.6.10 (Barwick's theorem of the heart, [Win24, Thm. 10.10])**.** *Let $\mathcal{C}$ be stable with t–structure.*

(1) If the t–structure is bounded below, then both

$$\mathrm{K}(\mathcal{C}_{\geq 0}) \longrightarrow \mathrm{K}(\mathcal{C}) \qquad \mathrm{K}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathrm{K}(\mathcal{C}_{\leq 0})$$

are equivalences.

(2) If the t–structure is bounded, then $\mathrm{K}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathrm{K}(\mathcal{C})$ is an equivalence.

Proof. For part (1), note that by Proposition 5.5.15, the inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ exhibits $\mathcal{C} \simeq \operatorname{colim}(\mathcal{C}_{\geq 0} \xrightarrow{\Sigma} \mathcal{C}_{\geq 0} \xrightarrow{\Sigma} \cdots)$ since the t–structure was bounded below. Hence, the induced map $\mathrm{K}(\mathcal{C}_{\geq 0}) \to \mathrm{K}(\mathcal{C})$ is an equivalence by Proposition 5.6.5.

Next, for the inclusion $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}_{\leq 0}$, we claim that the sequence with vanishing composite

$$\mathrm{K}(\mathcal{C}^{\heartsuit}) \longrightarrow \mathrm{K}(\mathcal{C}_{\leq 0}) \xrightarrow{\mathrm{K}(\tau_{\leq -1})} \mathrm{K}(\mathcal{C}^{\max}_{\leq -1}) \tag{5.8}$$

is a cofibre sequence of connective spectra, where the superscript max indicates the Waldhausen structure in which all morphisms are cofibrations. Since $\tau_{\leq -1} \colon \mathcal{C}_{\leq 0} \to \mathcal{C}_{\leq -1}$ is a Dwyer–Kan localisation at the unit maps $x \to \tau_{\leq -1}x$, whose fibres are all in $\mathcal{C}^{\heartsuit}$, by Idea 5.6.9, it therefore suffices to show that the induced functor

$$|\mathrm{pr}_{\mathrm{S}_n(\mathcal{C}^{\heartsuit})}\mathrm{S}_n(\mathcal{C}_{\leq 0})| \longrightarrow \mathrm{Fun}([n], \mathcal{C}_{\leq -1})^{\simeq}$$

is an equivalence for all n. Since every morphism in $\mathcal{C}_{\leq -1}$ is a cofibration in $\mathcal{C}_{\leq 0}$ using the exact structure on $\mathcal{C}_{\leq 0}$ from Proposition 5.5.21, we have an induced adjunction

$$\begin{tikzcd} \mathrm{S}_n(\mathcal{C}_{\leq 0}) \arrow[r, shift left, "\tau_{\leq -1}"] & \mathrm{S}_n(\mathcal{C}^{\max}_{\leq -1}) \simeq \mathrm{Fun}([n], \mathcal{C}_{\leq -1}) \arrow[l, shift left, "\mathrm{incl}"] \end{tikzcd}$$

The counit of this adjunction is an equivalence since it was a Bousfield localisation. Moreover, as we have observed above, the unit map of this adjunction is a projection with fibre in $\mathrm{S}_n(\mathcal{C}^{\heartsuit})$ and hence is a morphism in $\mathrm{pr}_{\mathrm{S}_n(\mathcal{C}^{\heartsuit})}\mathrm{S}_n(\mathcal{C}_{\leq 0})$. Furthermore, restricted to the subcategory $\mathrm{pr}_{\mathrm{S}_n(\mathcal{C}^{\heartsuit})}\mathrm{S}_n(\mathcal{C}_{\leq 0}) \subset \mathrm{S}_n(\mathcal{C}_{\leq 0})$, the functor $\tau_{\leq -1}$ sends every morphism to an equivalence. Hence, this adjunction restricts to an adjunction

$$\begin{tikzcd} \mathrm{pr}_{\mathrm{S}_n(\mathcal{C}^{\heartsuit})}\mathrm{S}_n(\mathcal{C}_{\leq 0}) \arrow[r, shift left, "\tau_{\leq -1}"] & \mathrm{Fun}([n], \mathcal{C}_{\leq -1})^{\simeq} \arrow[l, shift left, "\mathrm{incl}"] \end{tikzcd}$$

So applying $|-|$ to this adjunction concludes the proof of the claim.

By running the same argument as above, we also obtain a cofibre sequence of spectra

$$\mathrm{K}(\mathcal{C}_{\geq 0}) \longrightarrow \mathrm{K}(\mathcal{C}) \xrightarrow{\mathrm{K}(\tau_{\leq -1})} \mathrm{K}(\mathcal{C}^{\max}_{\leq -1}).$$

And so since $\mathrm{K}(\mathcal{C}_{\geq 0}) \to \mathrm{K}(\mathcal{C})$ was an equivalence by the first part, we see that $\mathrm{K}(\mathcal{C}^{\max}_{\leq -1}) \simeq 0$. Thus, by the cofibre sequence (5.8), we see that $\mathrm{K}(\mathcal{C}^{\heartsuit}) \to \mathrm{K}(\mathcal{C}_{\leq 0})$ is an equivalence, as wanted.

Finally, for part (2), if the t–structure is also bounded above, then the t–structure $(\mathcal{C}^{\mathrm{op}}_{\leq 0}, \mathcal{C}^{\mathrm{op}}_{\geq 0})$ from Exercise 5.5.6 on the stable category $\mathcal{C}^{\mathrm{op}}$ is bounded below. Hence, we may apply part (1) and use Observation 5.2.12 to get

$$\mathrm{K}(\mathcal{C}^{\heartsuit}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C}_{\leq 0}) \simeq \mathrm{K}(\mathcal{C}^{\mathrm{op}}_{\geq 0}) \xrightarrow{\simeq} \mathrm{K}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{K}(\mathcal{C})$$

This concludes the proof of the theorem. □

5.7 Interlude: coherence and regularity of rings

In this section, we study homological finiteness properties on rings and modules in the form of *almost perfectness, coherence,* and *regularity.* The material on the first two adjectives are based on Lurie's discussion in [Lur17, §7.2.4], whereas the part about regularity is based mainly on Barwick and Lawson's short and nice article [BL14]. This will form the foundations for our discussions on the (recent) applications of the theorem of the heart in the next section.

Almost perfectness

Exercise 5.7.1. Let R be a discrete associative ring. Show that a left R–module M is compact in the 1–category of left R–modules if and only if it is finitely presented.

Lemma 5.7.2 ([Lur17, Cor. 7.2.4.5 (1)])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ *and* $M \in \mathrm{LMod}(R)$ *perfect. Then* $\pi_n M = 0$ *for all* $n \ll 0$*.*

Proof. We have an equivalence $M \simeq \mathrm{colim}_n(\tau_{\geq -n}M)$. Since M is compact, it is a retract of some $\tau_{\geq -n}M$, so that $\pi_m M = 0$ for all $m \leq -n$. □

Definition 5.7.3. Let $\mathcal{C}$ be a compactly generated category. We say that an object $C \in \mathcal{C}$ is *almost compact* if $\tau_{\leq n}C$ is compact in $\tau_{\leq n}\mathcal{C}$ for all $n \geq 0$.

Lemma 5.7.4 ([Lur17, Rmk. 7.2.4.9])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$*. Then if* $M \in \mathrm{LMod}(R)$ *is compact, then it is almost compact.*

Proof. Since homotopy groups commute with filtered colimits, we see that the inclusion $\mathrm{LMod}(R)_{\leq n} \subseteq \mathrm{LMod}(R)$ is closed under filtered colimits. Hence, the truncation functor $\tau_{\leq n}\colon \mathrm{LMod}(R) \to \mathrm{LMod}(R)_{\leq n}$ preserves compact objects. □

Definition 5.7.5 ([Lur17, Def. 7.2.4.10])**.** Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$. We say that an $M \in \mathrm{LMod}(R)$ is *almost perfect* if there exists some k such that $M \in \mathrm{LMod}(R)_{\geq k}$ and M is almost compact as an object of $\mathrm{LMod}(R)_{\geq k}$. We write $\mathrm{LMod}(R)^{\mathrm{aperf}} \subseteq \mathrm{LMod}(R)$ for the full subcategory of almost perfect left R–modules.

Observation 5.7.6. It is clear that $\mathrm{LMod}(R)^{\mathrm{aperf}} \subseteq \mathrm{LMod}(R)$ is a thick subcategory.

Proposition 5.7.7 ([Lur17, Prop. 7.2.4.11 (3)])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ *and* $M \in \mathrm{LMod}(R)$*. If* M *is perfect, then it is almost perfect.*

Proof. Since M is perfect, it is a compact object in $\mathrm{LMod}(R)$, and by Lemma 5.7.2, we know that $M \in \mathrm{LMod}(R)_{\geq k}$. Fix an $n \geq 0$. Since the vertical inclusions in

$$\begin{tikzcd}
\mathrm{LMod}(R) & \mathrm{LMod}(R)_{\leq n} \arrow[l, hook'] \\
\mathrm{LMod}(R)_{\geq k} \arrow[u, hook] & \mathrm{LMod}(R)_{\geq k, \leq n}, \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

preserve filtered colimits, we see that an object in $\mathrm{LMod}(R)_{\geq k}$ (resp. $\mathrm{LMod}(R)_{\geq k, \leq n}$) is compact if it is compact in $\mathrm{LMod}(R)$ (resp. $\mathrm{LMod}(R)_{\leq n}$). Furthermore, note that the square obtained by taking horizontal left adjoints $\tau_{\leq n}$ also commutes. Hence, since by Lemma 5.7.4 we know that $\tau_{\leq n} M \in \mathrm{LMod}(R)_{\leq n}$ is compact, we see that it is also compact in $\mathrm{LMod}(R)_{\geq k, \leq n}$ as was to be shown. ☐

Definition 5.7.8. Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$. We say that a $M \in \mathrm{LMod}(R)$ has *Tor–amplitude* $\leq n$ if for every discrete right R–module N, the groups $\pi_i(N \otimes_R M)$ vanish for $i > n$. We say that M has *finite Tor–amplitude* if it has Tor–amplitude $\leq n$ for some n.

Exercise 5.7.9 (Yoga of Tor–amplitudes, [Lur17, Prop. 7.2.4.23])**.** Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$.

(1) If M has Tor–amplitude $\leq n$, then $\Sigma^k M$ has Tor–amplitude $\leq n + k$.

(2) Let $M' \to M \to M''$ be a fibre sequence in $\mathrm{LMod}(R)$. If M' and M'' have Tor–amplitudes $\leq n$, then so does M.

(3) Let M have Tor–amplitude $\leq n$. Then any retract of M has Tor–amplitude $\leq n$.

(4) Hence, conclude that the collection of left R–modules of finite Tor–amplitude is stable under retracts and finite colimits, and contains R, i.e. it is a thick subcategory containing R.

(5) Let $M \in \mathrm{LMod}(R)$ have finite Tor–amplitude and B a bounded below R–bimodule which has finite Tor–amplitude as a left R–module. Show that $B \otimes_R M$ also has finite Tor–amplitude as a left R–module. **Hint:** *note that* $N \otimes_R (B \otimes_R M) \simeq (N \otimes_R B) \otimes_R M$ *and induct on the number of homotopy groups of* $N \otimes_R B$ *by way of (2).*

Proposition 5.7.10 ([Lur17, Prop. 7.2.4.23 (4)])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ *and let* $M \in \mathrm{LMod}(R)$ *be almost perfect. Then* M *is perfect if and only if* M *has finite Tor–amplitude.*

Proof. The only if direction is an immediate consequence of Exercise 5.7.9 (4). For the converse, suppose M is almost perfect and of finite Tor–amplitude. Since M was almost perfect, we know that $M \in \mathrm{LMod}(R)_{\geq k}$ for some k, by definition, and so by Exercise 5.7.9 (1), we may assume that M was connective. We will prove by induction on the Tor–amplitude n, where the case $n = 0$ is given by [Lur17, Prop. 7.2.4.20].

Now since M is almost perfect, by using that $\pi_0 M$ is a finitely presented left $\pi_0 R$–module from Exercise 5.7.1, there exists a free left R–module P of finite rank and a fibre sequence

$$M' \longrightarrow P \longrightarrow M$$

where $P \to M$ is π_0–surjective. Since P was free of finite rank, it is perfect and so it suffices to argue that M' is perfect. To this end, we will show that M' has

Tor–amplitude $\leq n-1$, since this would imply that M' is perfect by the induction hypothesis.

For this, we wish to show that $\pi_k(N \otimes_R M') = 0$ for $k \geq n$. Consider the exact sequence of homotopy groups

$$\pi_{k+1}(N \otimes_R M) \longrightarrow \pi_k(N \otimes_R M') \longrightarrow \pi_k(N \otimes_R P).$$

The first term vanishes since M has Tor–amplitude $\leq n$. On the other hand, since P was free of finite rank, we know that $\pi_k(N \otimes_R P)$ is given by a finite sum of copies of $\pi_k N$, which are all zero since $k \geq n > 0$ and N was in the heart. Hence, $\pi_k(N \otimes_R M') = 0$ for $k \geq n$ as was to be shown. □

Coherence

Definition 5.7.11. Let R be an associative ring. We say that it is *left coherent* if every finitely generated left ideal of R is finitely presented as a left R–module.

Definition 5.7.12. Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$. We say that R is *left coherent* if $\pi_0 R$ is left coherent in the sense of Definition 5.7.11 and for each $n \geq 0$, the homotopy group $\pi_n R$ is a finitely presented left $\pi_0 R$–module.

Definition 5.7.13 ([BL14, Def. 1.1])**.** Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be left coherent and $M \in \mathrm{LMod}(R)$. We say that M is *truncated* if $\pi_m M = 0$ for $m \gg 0$. We say that M is *coherent* if it is both truncated and almost perfect. We write $\mathrm{Coh}(R) \subseteq \mathrm{LMod}(R)$ for the full subcategory of coherent modules.

Exercise 5.7.14 (Finitely presented is closed under extensions)**.** Let R be a discrete associative ring. Suppose we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ of left R–modules. Show that:

(1) If M' and M'' are finitely generated, then M is finitely generated too.

(2) If M' and M'' are finitely presented, then M is finitely presented too.

Hint: *use the snake lemma to prove (1), and then use the snake lemma again to prove (2) with (1) as an input.*

Exercise 5.7.15 ([Lur17, Lem. 7.2.4.15])**.** Let R be a left coherent discrete associative ring. Show that:

(1) Every finitely generated submodule of $R^{\oplus n}$ is finitely presented,

(2) Every finitely generated submodule of a finitely presented left R–module is finitely presented,

(3) If $f\colon M \to N$ is a map of finitely presented left R–modules, then $\ker(f)$ and $\mathrm{coker}(f)$ are finitely presented.

Proposition 5.7.16 ([Lur17, Prop. 7.2.4.17]). *Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be left coherent and $M \in \mathrm{LMod}(R)$. Then M is almost perfect if and only if M is bounded below and for all n, $\pi_n M$ is a finitely presented left $\pi_0 R$–module.*

Proof. We will only show the "only if" direction. Without loss of generality, suppose M is connective. We prove this by induction on n, where the base case of $n = 0$ is given Exercise 5.7.1. Given the base case, choose a map $\alpha\colon P \to M$ where P is a free R–module of finite rank and α is a π_0–surjection, and consider the fibre sequence

$$K \longrightarrow P \xrightarrow{\alpha} M.$$

Since α was π_0–surjective, K is connective, and by Observation 5.7.6, K is almost perfect. Thus, by induction, we know that $\pi_i K$ is finitely presented for $0 \leq i < n$.

Now consider the short exact sequence

$$0 \longrightarrow \mathrm{coker}(\pi_n K \to \pi_n P) \longrightarrow \pi_n M \longrightarrow \ker(\pi_{n-1} K \to \pi_{n-1} P) \longrightarrow 0.$$

The first term is finitely generated since it admits a surjection from $\pi_n P$, a finitely generated $\pi_0 R$–module. The last term is finitely presented by the inductive hypothesis and Exercise 5.7.15 (3). Hence, the middle term $\pi_n M$ is finitely generated. By the same reasoning again but with M replaced with K, we see that $\pi_n K$ is finitely generated too, so that $\mathrm{coker}(\pi_n K \to \pi_n P)$ is finitely presented. Therefore, by the short exact sequence above and Exercise 5.7.14, we see that $\pi_n M$ is finitely presented, as desired. ☐

Corollary 5.7.17 ([Lur17, Prop. 7.2.4.18]). *Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be left coherent. Then the t–structure on $\mathrm{LMod}(R)$ restricts to a t–structure $(\mathrm{LMod}(R)^{\mathrm{aperf}} \cap \mathrm{LMod}(R)_{\geq 0}, \mathrm{LMod}(R)^{\mathrm{aperf}} \cap \mathrm{LMod}(R)_{\leq 0})$ on $\mathrm{LMod}(R)^{\mathrm{aperf}}$. In this case, we have an identification of $(\mathrm{LMod}(R)^{\mathrm{aperf}})^{\heartsuit}$ with the category of finitely presented left modules over $\pi_0 R$.*

Proof. Since the functor $\tau_{\leq 0}$ clearly preserves perfect objects by definition, we only have to check that if M is almost perfect, then $\tau_{\geq 0} M$ is almost perfect. But this is clear from the characterisation of almost perfectness in terms of the homotopy groups from Proposition 5.7.16. The final statement about the heart is also an immediate consequence of Proposition 5.7.16. ☐

Regularity

Definition 5.7.18. Let R be a discrete left coherent ring. We say that it is *regular* if all finitely presented discrete left R–module M admits a finite length projective resolution.

Observation 5.7.19. Let R be a discrete left regular coherent ring and M a discrete finitely presented left R–module. Suppose N is a discrete right R–module. Then $\pi_i(N \otimes_R M) \cong \mathrm{Tor}_i(N, M)$, and so since M has a finite length projective resolution, this group vanishes for $i \gg 0$. That is, M has finite Tor–amplitude.

The following definition was given in [BL14, Def. 1.2, Def. 14].

Definition 5.7.20. Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be left coherent. We say that it is:

- *almost regular* if any coherent left R–module has finite Tor–amplitude. Equivalently by Proposition 5.7.10, every coherent left R–module is perfect,
- *regular* if $\pi_0 R$ is a regular ring (in the sense of Definition 5.7.18), and $\pi_0 R$ has finite Tor–amplitude as an R–module.

Proposition 5.7.21 ([BL14, Prop. 1.3])**.** *Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be left coherent. If it is regular, then it is almost regular.*

Proof. Since the perfect objects form a thick subcategory, it suffices to prove that an almost perfect module M in the heart is perfect. That is, by Proposition 5.7.16, we need to show that for any finitely presented discrete left $\pi_0 R$–module M, we have that M has finite Tor–amplitude over R. Since $\pi_0 R$ is a regular discrete ring, we know by Observation 5.7.19 that M has finite Tor–amplitude over $\pi_0 R$. Now, let $N \in \mathrm{RMod}(R)$ be discrete. Since $\pi_0 R$ has finite Tor–amplitude as an R–module, we get that $N \otimes_R \pi_0 R$ has finitely many homotopy groups. On the other hand, we have $N \otimes_R M \simeq (N \otimes_R \pi_0 R) \otimes_{\pi_0 R} M$. Hence, by induction on the number of homotopy groups of $N \otimes_R \pi_0 R$ and using that M has finite Tor–amplitude over $\pi_0 R$, we obtain that $N \otimes_R M$ has finitely many homotopy groups, as required. ☐

Proposition 5.7.22 (Criterion for regularity, [BL14, Prop. 1.5])**.** *Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be left coherent and $\pi_0 R$ be regular. Then R is regular if and only if the left $\pi_0 R$–module $\pi_0 R \otimes_R \pi_0 R$ has finite Tor–amplitude.*

Proof. The "only if" direction is immediate by Exercise 5.7.9 (5). The converse is also immediate from the observation that $N \otimes_R \pi_0 R \simeq N \otimes_{\pi_0 R} (\pi_0 R \otimes_R \pi_0 R)$. ☐

Next, we record the following oft–used result (cf. also [BL22, Prop. 2.4]).

Proposition 5.7.23. *Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be left almost regular coherent. Then the t–structure on $\mathrm{LMod}(R)$ restricts to a t–structure $(\mathrm{Perf}(R) \cap \mathrm{LMod}(R)_{\geq 0}, \mathrm{Perf}(R) \cap \mathrm{LMod}(R)_{\leq 0})$ on $\mathrm{Perf}(R)$. In this case, we have an identification of $\mathrm{Perf}(R)^{\heartsuit}$ with the category of finitely presented left modules over $\pi_0 R$.*

Proof. By the argument as in the proof of Corollary 5.7.17, we are left to showing that if M is perfect, then $\tau_{\geq 0} M$ is perfect. By the aforementioned proof, we know at least that $\tau_{\geq 0} M$ is almost perfect. Therefore, by Proposition 5.7.10, if we can show that M has finite Tor–amplitude, then we would get that M is perfect as desired.

To see this, since M was compact, we know that there exists a k such that $M \in \mathrm{LMod}(R)_{\geq k}$. We will prove by induction on $|k|$ that $\tau_{\geq 0} M$ has finite Tor–amplitude. If $k \geq 0$, then $\tau_{\geq 0} M \simeq M$, and so there is nothing to prove. Suppose $k < 0$. Define C to be the cofibre in

$$\tau_{\geq k} M \longrightarrow \tau_{\geq k-1} M \longrightarrow C.$$

Now, C is concentrated in one degree and is almost perfect. Hence, by almost regularity, we know that C has finite Tor–amplitude over R. On the other hand, by induction, we know that $\tau_{\geq k}M$ has finite Tor–amplitude. Hence, from the fibre sequence and Exercise 5.7.9 (2), we see that $\tau_{\geq k-1}M$ also has finite Tor–amplitude, completing the induction. Finally, the statement about the heart is immediate from the characterisation of Exercise 5.7.1. □

5.8 Blumberg–Mandell regularity localisation sequence

Our goal in this section is to prove Barwick's formulation of a celebrated localisation sequence of Blumberg and Mandell [BM08]. For this section and the next, we will need to the following fantastic recent result of Antieau–Gepner–Heller, which is heavily built upon the ideas of Schlichting. We will unfortunately not have the space and time to prove it in this course.

Theorem 5.8.1 ([AGH18, Thm. 1.1, Thm. 1.2])**.** *If $\mathcal{E} \in \mathrm{Cat}^{\mathrm{ex}}$ admits a bounded t–structure, then $\mathbb{K}_{-1}(\mathcal{E}) = 0$. Moreover, if $\mathcal{E}^{\heartsuit}$ is Noetherian, then $\mathbb{K}_{-n}(\mathcal{E}) = 0$ for all $n \geq 1$.*

Recall first by construction of nonconnective K–theory from Theorem 4.5.46 that for an idempotent–complete small stable category $\mathcal{C}$, $\Omega^{\infty}\mathbb{K}(\mathcal{C}) \simeq \mathcal{K}(\mathcal{C})$, and hence $\mathrm{K}(\mathcal{C}) \simeq \tau_{\geq 0}\mathbb{K}(\mathcal{C})$. We will also need the following piece of language.

Terminology 5.8.2. Let $R \in \mathrm{Alg}(\mathrm{Sp})$ and $R \to L$ a map of $\mathbb{E}_1$–algebras. We say that L is a *smashing R–algebra* if the left adjoint basechange functor

$$L \otimes_R - \colon \mathrm{LMod}(R) \longrightarrow \mathrm{LMod}(L)$$

is a Bousfield localisation, i.e. its right adjoint is fully faithful. Additionally, we say that the smashing R–algebra L *has compact fibre* if the fibre $\mathrm{LMod}(R)^{L-\mathrm{tors}}$ of the Bousfield localisation $L \otimes_R -$ is generated by compact objects and the inclusion $\mathrm{LMod}(R)^{L-\mathrm{tors}} \subseteq \mathrm{LMod}(R)$ preserves compact objects.

We are now ready to state and prove the main result of this section.

Theorem 5.8.3 (Regularity localisation sequence, [Bar15, Thm. 8.8])**.** *Let $R \in \mathrm{Alg}(\mathrm{Sp})_{\geq 0}$ be a left regular coherent ring and L a smashing R–algebra with compact fibre*[1]*. Suppose that a perfect left R–module is L–acyclic if and only if it is truncated. Then there is a fibre sequence of connective spectra*

$$\mathrm{K}(\pi_0 R) \longrightarrow \mathrm{K}(R) \longrightarrow \mathrm{K}(L)$$

---

[1] Apparently, [Bar15, Thm. 8.7] says that this is automatic but I have not been able to figure out why. Also, [BL14, Thm. 2.1] used a slightly different hypothesis, which was what I followed in this theorem. I have a feeling that the hypothesis used in [Bar15, Thm. 8.7] is too strong, since one can take infinite direct sums of truncated acyclic objects, which is no longer acyclic.

Proof. By the compact generation hypothesis, we may apply $(-)^\omega$ to the Verdier sequence of large categories $\mathrm{LMod}(R)^{L-\mathrm{tors}} \hookrightarrow \mathrm{LMod}(R) \to \mathrm{LMod}(L)$ to obtain via Theorem 4.5.35 the Karoubi sequence $\mathrm{Perf}(R)^{L-\mathrm{tors}} \to \mathrm{Perf}(R) \to \mathrm{Perf}(L)$ of idempotent–complete stable categories. Hence, by Theorem 4.5.46, we get a fibre sequence

$$\mathbb{K}(\mathrm{Perf}(R)^{L-\mathrm{tors}}) \longrightarrow \mathbb{K}(R) \longrightarrow \mathbb{K}(L)$$

of spectra. Applying $\tau_{\geq 0}$ gives a fibre sequence in CGrp

$$\mathrm{K}(\mathrm{Perf}(R)^{L-\mathrm{tors}}) \longrightarrow \mathrm{K}(R) \longrightarrow \mathrm{K}(L). \tag{5.9}$$

By Proposition 5.7.23, we have an induced t–structure on $\mathrm{Perf}(R)$. On the other hand, by our hypothesis that perfect L–acyclic objects are precisely the truncated ones and by definition of almost regularity, we have an identification of $\mathrm{Perf}(R)^{L-\mathrm{tors}}$ with $\mathrm{Coh}(R)$ (c.f. Definition 5.7.13). Furthermore, it is then clear that the inclusion $\mathrm{Coh}(R) \subseteq \mathrm{Perf}(R)$ under this identification restricts the t–structure on $\mathrm{Perf}(R)$ to a bounded on $\mathrm{Coh}(R)$. Now, by Proposition 5.7.16, we know that $\mathrm{Coh}(R)^\heartsuit$ is identified with the category $\mathrm{FPres}(\pi_0 R)$ of finitely presented discrete $\pi_0 R$–modules.

On the other hand, since $\pi_0 R$ is left regular coherent, we know by Proposition 5.7.23 that the t–structure on $\mathrm{LMod}(\pi_0 R)$ restricts to one on $\mathrm{Perf}(\pi_0 R)$, which must necessarily be bounded since $\pi_0 R$ was connective and coconnective. By Exercise 5.7.1, we learn that $\mathrm{Perf}(\pi_0 R)^\heartsuit$ is identified with $\mathrm{FPres}(\pi_0 R)$.

All in all, by two applications of Theorem 5.6.10, we identify $\mathrm{K}(\mathrm{Perf}(R)^{L-\mathrm{tors}})$ as

$$\mathrm{K}(\mathrm{Coh}(R)) \simeq \mathrm{K}(\mathrm{Coh}(R)^\heartsuit) \simeq \mathrm{K}(\mathrm{FPres}(\pi_0 R)) \simeq \mathrm{K}(\mathrm{Perf}(\pi_0 R)^\heartsuit) \simeq \mathrm{K}(\pi_0 R).$$

Furthermore, by Theorem 5.8.1, we know that $\mathbb{K}_{-1}(\mathrm{Coh}(R)) = 0$. Hence, by Exercise 4.4.7, the sequence (5.9) yields the required fibre sequence in spectra. □

Lemma 5.8.4. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})_{\geq 0}$ such that there exists an $n > 0$ and a class $\beta \in \pi_n R$ such that the induced map $\beta\colon \pi_m R \to \pi_{m+n} R$ is an isomorphism for $m \gg 0$. Let M be a perfect R–module. Then $\beta\colon \pi_k M \to \pi_{k+n} M$ is an isomorphism for $k \gg 0$.*

Proof. The property of interest is satisfied by R, and so if we could show that the subcategory of objects satisfying the property is a thick subcategory, then we will have shown that all perfect modules satisfy it. It is clearly preserved by taking (de)suspensions and retracts and so we are left to show that this property is closed under taking cofibres, i.e. if $A \to B \to C$ is a cofibre sequence with A and B satisfying said property, then so does C. Without loss of generality, suppose $r \gg 0$ such that $\beta\colon \pi_k X \to \pi_{k+n} X$ is an isomorphism for all $k \geq r$ when $X \in \{A, B\}$. Then by the long exact sequences of homotopy groups associated to the map of cofibre sequences

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\beta"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[d, "\beta"'] \\
\Sigma^{-n} A \arrow[r] & \Sigma^{-n} B \arrow[r] & \Sigma^{-n} C
\end{tikzcd}$$

we see that $\beta\colon \pi_k C \to \pi_{k+n} C$ is an isomorphism for all $k \geq r+1$, as desired. □

Corollary 5.8.5. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})_{\geq 0}$ such that there exists an $n > 0$ and a map $\beta\colon \Sigma^n R \to R$ of R–modules which is an equivalence. Let M be a perfect R–module. Then it is $R[\beta^{-1}]$–acyclic if and only if it is truncated.*

Proof. Suppose M is truncated. To see that $R[\beta^{-1}] \otimes_R M \simeq 0$, we show that $\pi_k(R[\beta^{-1}] \otimes_R M) = 0$ for all k. Fixing a k, we note that

$$\pi_k(R[\beta^{-1}] \otimes_R M) \cong \operatorname*{colim}_{\mathbb{N}}(\pi_k M \xrightarrow{\beta} \pi_{k+n} M \xrightarrow{\beta} \pi_{k+2n} M \xrightarrow{\beta} \cdots).$$

Hence, since M was truncated, $\pi_{k+mn} M = 0$ for $m \gg 0$, and so $\pi_k(R[\beta^{-1}] \otimes_R M) = 0$ as required. For the converse, suppose for contradiction that M is not truncated. By Lemma 5.8.4, $\beta\colon \pi_k M \to \pi_{k+n} M$ is an isomorphism for $k \gg 0$. Since M is not truncated, let $\ell \gg 0$ be such such that $\pi_\ell M \neq 0$ and $\beta\colon \pi_k M \to \pi_{k+n} M$ for all $k \geq \ell$. Then we get

$$0 = \pi_\ell(R[\beta^{-1}] \otimes_R M) = \operatorname*{colim}_{\mathbb{N}}(\pi_\ell M \xrightarrow[\cong]{\beta} \pi_{\ell+n} M \xrightarrow[\cong]{\beta} \pi_{\ell+2n} M \xrightarrow[\cong]{\beta} \cdots) \cong \pi_\ell M \neq 0$$

which is a contradiction, as wanted. □

Recollections 5.8.6 (Topological K–theories)**.** There are the connective spectra of complex and real topological K–theories ku and ko respectively, which are defined as

$$\mathrm{ku} := (\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq})^{\mathrm{gp}} \quad \text{and} \quad \mathrm{ko} := (\mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.},\simeq})^{\mathrm{gp}}.$$

There are naturally endowed with the structure of $\mathbb{E}_\infty$–ring spectra by virtue of Construction 3.1.5 and using the tensor products to view $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.},\simeq}$ and $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{f.d.},\simeq}$ as objects in CAlg(CMon(Cat)). A famous fundamental result about these objects is the *Bott periodicity*, which states that they are 2–periodic and 8–periodic in nonnegative degrees, respectively. The homotopy groups look as follows:

$r \geq 0$	$2n$	$2n+1$
$\pi_r\mathrm{ku}$	$\mathbb{Z}$	0

$r \geq 0$	$8n$	$8n+1$	$8n+2$	$8n+3$	$8n+4$	$8n+5$	$8n+6$	$8n+7$
$\pi_r\mathrm{ko}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

Upon inverting the generators $\beta \in \pi_2\mathrm{ku}$ and $\beta^4 \in \pi_8\mathrm{ko}$,we obtain the $\mathbb{E}_\infty$–ring spectra

$$\mathrm{KU} := \mathrm{ku}[\beta^{-1}] \quad \text{and} \quad \mathrm{KO} := \mathrm{ko}[(\beta^4)^{-1}],$$

called the 2–periodic complex K–theory and 8–periodic real K–theory, respectively.

Theorem 5.8.7 ([BM08, Localization Theorem],[BL14, Cor. 3.2])**.** *The maps of* $\mathbb{E}_\infty$*–rings* $\mathrm{ku} \to \mathrm{KU}$ *and* $\mathrm{ko} \to \mathrm{KO}$ *induce fibre sequences of connective algebraic K–theory*

$$\mathrm{K}(\mathbb{Z}) \longrightarrow \mathrm{K}(\mathrm{ku}) \longrightarrow \mathrm{K}(\mathrm{KU}) \qquad \mathrm{K}(\mathbb{Z}) \longrightarrow \mathrm{K}(\mathrm{ko}) \longrightarrow \mathrm{K}(\mathrm{KO})$$

in Sp.

Proof. Since $\pi_0\mathrm{ku} \cong \mathbb{Z} \cong \pi_0\mathrm{ko}$, we are left to argue that the maps $\mathrm{ku} \to \mathrm{KU}$ and $\mathrm{ko} \to \mathrm{KO}$ satisfy the hypothesis of (5.9). We only do the case of ko since the case of ku is similar (and slightly easier). Since $\mathrm{KO} \simeq \mathrm{ko}[(\beta^4)^{-1}]$, we see that it is clearly smashing. Next, we show that it has compact fibres by showing that ko/β^4 is a compact generator and using Proposition 2.1.33. To do this, let $M \in \mathrm{Mod}(\mathrm{ko})^{\mathrm{KO-tors}}$. Suppose $\mathrm{map}(\mathrm{ko}/\beta^4, M) \simeq 0$. Hence, the map $\beta^4 \colon \mathrm{ko} \to \Sigma^{-8}\mathrm{ko}$ is an equivalence. But then this implies that $0 \simeq \mathrm{KO} \otimes_{\mathrm{ko}} M \simeq M[(\beta^4)^{-1}] \simeq M$. Hence, ko/β^4 is a compact generator, as wanted. Furthermore, by Corollary 5.8.5, we know that the perfect acyclics are precisely the truncated perfect ko–modules. Finally, we need to argue that ko is left almost regular coherent. That it is coherent and that $\pi_0\mathrm{ko} \cong \mathbb{Z}$ is regular are clear, and we use the criterion Proposition 5.7.22 to show that it is even a regular ring. By the cofibre sequences (the first of which is a well–known theorem of Wood's)

$$\Sigma\mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \longrightarrow \mathrm{ku} \qquad \Sigma^2\mathrm{ku} \xrightarrow{\beta} \mathrm{ku} \longrightarrow \mathbb{Z}$$

in Mod(ko), we see that $\mathbb{Z} = \pi_0\mathrm{ko}$ is a perfect ko–module, and so has finite Tor–amplitude over ko by Proposition 5.7.10. This completes the proof. ☐

5.9 Burklund–Levy unipotent devissage

This is the theory set up in [BL21]. For brevity, we will provide here a minimal account of their theory sufficient to state and prove the fundamental theorems. These results are best viewed in the language of *noncommutative geometry* of the two authors from [BL22] to which we refer the interested reader.

Notation 5.9.1. Let $\mathrm{Cat}^{\mathrm{perf}} \subseteq \mathrm{Cat}^{\mathrm{st}}$ be the full subcategory of idempotent–complete stable categories.

Definition 5.9.2 ([BL22, §1])**.** Let $\mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ be the category whose objects are pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$ where $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ full subcategory which is idempotent–complete and closed under finite colimits and extensions.

Construction 5.9.3. Since $\mathrm{Ind}(\mathcal{C})$ is presentable, we can apply Proposition 5.5.18 to obtain a t–structure on $\mathrm{Ind}(\mathcal{C})$ such that $\mathrm{Ind}(\mathcal{C})_{\geq 0} = \mathrm{Ind}(\mathcal{C}_{\geq 0})$ (note that this is not stable since we are not allowing loops!).

Definition 5.9.4 ([BL21, Def. 3.4])**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}_{\geq 0}$ and $\mathcal{D} \in \mathrm{Cat}^{\mathrm{perf}}$. We say that:

- $\mathcal{C}$ is *regular* if the t–structure from Construction 5.9.3 restricts to one on $\mathcal{C}$ (i.e. the truncation functors restrict to $\mathcal{C}$).
- $\mathcal{C}$ is *bounded* if each $c \in \mathcal{C}$ is bounded in the t–structure on $\mathrm{Ind}(\mathcal{C})$.
- an exact functor $\mathcal{C} \to \mathcal{D}$ is *quasi–affine* if its image generates $\mathcal{D}$ under finite colimits and retracts,
- a quasi–affine functor $\mathcal{C} \to \mathcal{D}$ is *unipotent* if it is fully faithful on $\mathcal{C}^{\heartsuit}$.

Notation 5.9.5. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of small stable categories. Then using the left Kan extension, we have an adjunction

$$F_!\colon \mathrm{Ind}(\mathcal{C}) \rightleftharpoons \mathrm{Ind}(\mathcal{D}) : F^*$$

where we have a commuting square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, hook] & \mathcal{D} \arrow[d, hook] \\
\mathrm{Ind}(\mathcal{C}) \arrow[r, "F_!"] & \mathrm{Ind}(\mathcal{D})
\end{tikzcd}$$

In general, checking the unipotence condition is hard, and before proceeding with the main content of this subsection, we provide a more usable criterion for this condition.

Lemma 5.9.6 ([BL21, Lem. 3.8])**.** *If for every* $c \in \mathcal{C}^{\heartsuit}$*,* $\mathrm{cofib}(c \to F^*F_!(c)) \in \mathrm{Ind}(\mathcal{C})_{\leq -1}$*, then the map* $F\colon \mathcal{C} \to \mathcal{D}$ *is unipotent.*

Proof. We only prove the if direction (as this is easier and is the more interesting direction anyway). Let $c, c' \in \mathcal{C}^{\heartsuit}$. We want to show that the map $F\colon \mathrm{Map}_{\mathcal{C}}(c,c') \to \mathrm{Map}_{\mathcal{D}}(Fc,Fc')$ is an equivalence. But then $\mathrm{Map}_{\mathcal{D}}(Fc,Fc') \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(F_!c,F_!c') \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(c,F^*F_!c')$, and so the map of interest sits in a fibre sequence

$$\mathrm{Map}_{\mathcal{C}}(c,c') \to \mathrm{Map}_{\mathcal{D}}(Fc,Fc') \longrightarrow \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(c,\mathrm{cofib}(c' \to F^*F_!c'))$$

where the last term is contractible by hypothesis, and so we are done. □

The following pair of Lemmas 5.9.7 and 5.9.11 are the key inputs given in the paper to descend bounded regularity along unipotent maps. The main idea is to construct the desired t–structure by enlarging via the Ind–completion, for which we have a good presentable theory for t–structures.

Lemma 5.9.7 ([BL21, Lem. 2.3])**.** *Let* $\mathcal{C}$ *be bounded and* $F\colon \mathcal{C} \to \mathcal{D}$ *be a unipotent map in* $\mathrm{Cat}^{\mathrm{perf}}$*. If we equip* $\mathrm{Ind}(\mathcal{D})$ *with the* t*–structure associated to* $F(\mathcal{C}_{\geq 0})$*, then* $F_!\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D})$ *is* t*–exact.*

Proof. By construction, $F_!$ sends connectives to connectives. To see that $F_!$ also preserves coconnectivity, note first that the connective part of the t–structure on $\mathrm{Ind}(\mathcal{D})$ is generated by $F(\mathcal{C}_{\geq 0})$ and t–structures are overdetermined by Lemma 5.5.9. Hence, we just need to show that for all $c \in \mathcal{C}_{\geq 1}$ and $x \in \mathrm{Ind}(\mathcal{C})_{\leq 0}$, the mapping space $\mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(F_!c, F_!x)$ is contractible. Moreover, since coconnectivity in $\mathrm{Ind}(\mathcal{C})$ is tested by mapping out of compact objects, the inclusion $\mathrm{Ind}(\mathcal{C})_{\leq 0} \subseteq \mathrm{Ind}(\mathcal{C})$ preserves filtered colimits, and so x can be written as a filtered colimit of compact coconnectives because the Bousfield localisation $\tau_{\leq 0} \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})_{\leq 0}$ preserves compact objects and everything in $\mathrm{Ind}(\mathcal{C})$ may be written as a filtered colimit of compacts. Since $F_!$ preserves colimits, it thus suffices to show that $\mathrm{Map}_{\mathrm{Ind}(\mathcal{D})}(F_!c, F_!x) \simeq *$ when x is compact, i.e. $x \in \mathcal{C}_{\leq 0}$. By boundedness of the t–structure on $\mathcal{C}$, we can reduce both variables to the case where c and x are both concentrated in one degree, so suppose without loss of generality that $c \in \mathcal{C}^{\heartsuit_n} \coloneqq \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq n}$ and $x \in \mathcal{C}^{\heartsuit}$. Now consider the map of fibre sequences

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(c, \Sigma^{k-1}x) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(c, 0) \arrow[r] \arrow[d, equal] & \mathrm{Map}_{\mathcal{C}}(c, \Sigma^{k}x) \arrow[d] \\
\mathrm{Map}_{\mathcal{D}}(Fc, F\Sigma^{k-1}x) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(Fc, F0) \arrow[r] & \mathrm{Map}_{\mathcal{D}}(Fc, F\Sigma^{k}x).
\end{tikzcd}$$

When $k = n$ we have an equivalence by the unipotence hypothesis, and hence by downward induction it is an equivalence also when $k = 0$. But then $\mathrm{Map}_{\mathcal{C}}(c, x) \simeq *$, and so we are done. ☐

Our next goal is to explain the key Lemma 5.9.11 of Burklund and Levy which says that unipotence guarantees a devissage situation. The arguments in therein are organised efficiently and beautifully with a noncomputational use of spectral sequences, and for this we will need three preliminary setups.

Proposition 5.9.8 (Filtration pasting for extensions)**.** *Let $\mathcal{B}$ be an abelian category and $\mathcal{A} \subseteq \mathcal{B}$ an exact inclusion of abelian categories. Suppose $A, C \in \mathcal{B}$ admit finite filtrations of lengths n and m, respectively, whose filtration quotients are in $\mathcal{A}$. Then for any extension $A \rightarrowtail B \twoheadrightarrow C$, B admits such a filtration also with length $n + m + 1$.*

Proof. Without loss of generality by induction, we may suppose that the filtration on C is of length 1, i.e. we have $0 \rightarrowtail C_1 \rightarrowtail C$ where C_1 and C/C_1 are both in $\mathcal{A}$. Writing $P_1 \coloneqq B \times_C C_2$ and $P_2 \coloneqq B \times_C 0 \cong A$, consider the solid part of the diagram

$$\begin{tikzcd}
 & P_2 \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d, tail] \\
 & P_1 \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, tail] \\
A \arrow[r, tail] \arrow[uur, dashed, "\cong"] \arrow[ur, dashed] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Since the right horizontal arrows are surjective, the pullbacks are also pushouts and so

$$B/P_1 \cong C/C_1 \quad \text{and} \quad P_1/P_2 \cong C_1$$

are both in $\mathcal{A}$. Now concatenating with the filtration on A we obtain the finite filtration

$$0 \rightarrowtail A_n \rightarrowtail \cdots \rightarrowtail A_1 \rightarrowtail A \cong P_2 \rightarrowtail P_1 \rightarrowtail B$$

with the required property on associated graded. □

Construction 5.9.9 (Filtration pasting for cofibres, [BL21, Proof of Prop. 2.4])**.** Let $A, B \in \mathcal{D}^{\heartsuit}$ be objects in the heart of a stable category equipped with finite filtrations and suppose $r \colon A \to B$ is a map. In this construction, we explain how to obtain a finite filtration on the cofibre $\mathrm{cofib}(r)$ in a slick way.

We first explain how to view r as a map on the filtrations. For $X \in \{A, B\}$, write $X_\bullet \colon \mathbb{Z} \to \mathcal{D}$ for the datum of the filtration, where $X_n = X$ and $X_{-n} = 0$ for $n \gg 0$. As such, note for example that the associated graded $\mathrm{gr}(X_\bullet)$ is nonzero in only finitely many degrees. Furthermore, since the filtrations on A and B are finite, we may shift the gradings os that whenever $\mathrm{gr}_i B$ and $\mathrm{gr}_j A$ are nonzero, then $j < i$. Using this, we may then canonically lift the map $r \colon A \to B$ to a map $r_\bullet \colon A_\bullet \to B_\bullet$, which schematically looks like

$$\begin{tikzcd}
\cdots 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A_{-m} \arrow[r] \arrow[d, "r"] & \cdots A_{-1} \arrow[r] \arrow[d, "r"] & A_0 = A \arrow[r, equal] \arrow[d, "r"] & A \cdots \arrow[d, "r"] \\
\cdots 0 \arrow[r] & \cdots \arrow[r] & B_{-1} \arrow[r] & B_0 = B \arrow[r, equal] & \cdots B \arrow[r, equal] & B \arrow[r, equal] & B \cdots
\end{tikzcd}$$

By may now obtain the desired finite filtration on $\mathrm{cofib}(r)$ by considering $\mathrm{cofib}(r_\bullet) \in \mathrm{Fun}(\mathbb{Z}, \mathcal{D})$.

Next, we recall that we have spectral sequences associated to filtered objects.

Recollections 5.9.10 (Spectral sequences from filtered objects)**.** Suppose we are given a filtered object $X_\bullet \in \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$

$$\cdots \xrightarrow{f_{-1}} X_{-1} \xrightarrow{f_0} X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots$$

in a stable category $\mathcal{C}$ equipped with a t–structure. Lurie then constructs in [Lur17, §1.2.2] a spectral sequence

$$E_1^{p,q} = \pi_{p+q}\,\mathrm{cofib}(f_p) \in \mathcal{C}^{\heartsuit}$$

which converges to $\pi_* \operatorname{colim}_n X_n$ for example when $X_k = 0$ for $k \ll 0$ (c.f. [Lur17, Prop. 1.2.2.14]). For example, when we have a finite filtration $B_\bullet$ of an object B, as in Construction 5.9.9, we obtain a spectral sequence whose E_1–page is given by the homotopy objects of the associated gradeds converging to $\pi_* B$.

We are now ready to state and prove the key lemma.

Lemma 5.9.11 (Unipotence guarantees devissage, [BL21, Prop. 2.4])**.** *In the situation of Lemma 5.9.7, the induced t–structure on* $\mathrm{Ind}(\mathcal{D})$ *restricts to a bounded t–structure on* $\mathcal{D}$*. Moreover, each* $d \in \mathcal{D}^{\heartsuit}$ *has a finite filtration with associated graded in the image of* $F|_{\mathcal{C}^{\heartsuit}}$*.*

Proof. Let $\mathcal{E} \subseteq \mathcal{D}$ be the subcategory of objects $d \in \mathcal{D}$ such that d is bounded in the t–structure on $\mathrm{Ind}(\mathcal{D})$ and $\pi_i^{\heartsuit}(d)$ admits a finite filtration with quotients in $F(\mathcal{C}^{\heartsuit})$. Clearly, $F(\mathcal{C}) \subseteq \mathcal{E}$. Note that since $\mathcal{C}^{\heartsuit}$ has compact image in $\mathrm{Ind}(\mathcal{D})$, each such $\pi_i^{\heartsuit}(d)$ and d itself will be compact also. Hence, if we can show that $\mathcal{E} \subseteq \mathcal{D}$ is in fact all of $\mathcal{D}$, we would have simultaneously shown the two statements in the proposition since this would say that the t–structure truncations on $\mathrm{Ind}(\mathcal{D})$ restrict to functors on $\mathcal{D}$, i.e. that $\mathcal{D}$ inherits the t–structure from $\mathrm{Ind}(\mathcal{D})$, and moreover that this is bounded. Thus, by quasi–affineness, it would suffice to show that $\mathcal{E} \subseteq \mathcal{D}$ is thick.

Here is a trick observation: taking finite colimits and retracts only effect taking kernels, cokernels, and extensions on homotopy groups (for instance, for a retract a of a $d \in \mathcal{E}$, we obtain the associated idempotent map $e\colon d \to d$ such that on $\pi_i^{\heartsuit}$, $\mathrm{coker}(\ker(e))$ yields $\pi_i^{\heartsuit} a)$. Hence, since the conditions in the definition of $\mathcal{E}$ are stated purely in terms of homotopy groups, it would now suffice to show that the condition

$$d \in \mathcal{D}^{\heartsuit} \text{ has a finite filtration whose associated gradeds lie in } F(\mathcal{C}^{\heartsuit}). \tag{5.10}$$

is closed under taking kernels, cokernels, and extensions, since this would mean that $\mathcal{E}$ is stable under taking finite colimits and retracts, i.e. thick. By Proposition 5.9.8, we may paste filtrations to handle the case of extensions, so we are left to deal with kernels and cokernels. These cases will be dealt with simultaneously.

Let $r\colon A \to B$ be a map in $\mathrm{Ind}(\mathcal{D})^{\heartsuit}$ where A and B have filtrations of the required form. Let $C = \mathrm{cofib}(r)$ be equipped with the finite filtration $C_\bullet$ from Construction 5.9.9. Note that by construction, this has associated gradeds are given first (counting from the left) by $\mathrm{gr}(B_\bullet)$ and the by $\Sigma\mathrm{gr}(A_\bullet)$. By the spectral sequence construction Recollection 5.9.10 applied to $C_\bullet$, we obtain a spectral sequence converging to the associated gradeds of a finite filtration on $\pi_* C$ whose E_1–page is given by the π_* of $\mathrm{gr}(B_\bullet)$ and $\Sigma\mathrm{gr}(A_\bullet)$, which are objects in $\mathcal{C}^{\heartsuit}$ by hypothesis.

Now, since F is t–exact and fully faithful on $\mathcal{C}^{\heartsuit}$, kernels and cokernels of maps between objects in the image of $F|_{\mathcal{C}^{\heartsuit}}$ remain in the image of $F|_{\mathcal{C}^{\heartsuit}}$. Thus, as we turn the page of the spectral sequence by computing kernels and cokernels from the differentials, the terms remain in the image of $F|_{\mathcal{C}^{\heartsuit}}$. Since the filtration was finite, the spectral sequence terminates in a finite amount of steps to yield an E_∞–page with objects in the image of $F|_{\mathcal{C}^{\heartsuit}}$. This gives a finite filtration of the desired type on $\ker(r) \simeq \pi_1^{\heartsuit}\,\mathrm{cofib}(r)$ and $\mathrm{coker}(r) \simeq \pi_0^{\heartsuit}\,\mathrm{cofib}(r)$ as required. □

We can now collect and amplify these results in the following, which is the main theorem of [BL21].

Theorem 5.9.12 (Burklund–Levy unipotent devissage, [BL21, Thm. 1.3])**.** *Let $\mathcal{C}$ be bounded and $F\colon \mathcal{C} \to \mathcal{D}$ be a unipotent map in* $\mathrm{Cat}^{\mathrm{perf}}$*. Then there is a correspoing bounded t–structure on $\mathcal{D}$ for which F is t–exact. Moreover, the induced maps on connective K–theory*

$$\begin{tikzcd}
\mathrm{K}(\mathcal{C}^\heartsuit) \arrow[r] \arrow[d] & \mathrm{K}(\mathcal{D}^\heartsuit) \arrow[d] \\
\mathrm{K}(\mathcal{C}) \arrow[r] & \mathrm{K}(\mathcal{D})
\end{tikzcd}$$

are all equivalences. Additionally, $\mathbb{K}_{-1}(\mathcal{C}) = 0 = \mathbb{K}_{-1}(\mathcal{D})$.

Proof. By Lemmas 5.9.7 and 5.9.11 we know all the statements about t–structures. By Theorem 5.8.1, we then have also that $\mathbb{K}_{-1}(\mathcal{C}) = 0 = \mathbb{K}_{-1}(\mathcal{D})$. Hence, we are left to prove the statement about the equivalences in the square. By Barwick's Theorem 5.6.10 the vertical maps are equivalences, and so we only need to show that the top horizontal map is an equivalence. For this, we want to apply Quillen's Theorem 5.4.1. By unipotence, $\mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is fully faithful, and Lemma 5.9.11 already shows the required filtration condition. Thus, it remains to check that if $d \in \mathcal{D}^\heartsuit$ is a subobject of $c \in \mathcal{C}^\heartsuit$, then $d \in \mathcal{C}^\heartsuit$. By exactness of the inclusion, it suffices to show that $c/d \in \mathcal{C}^\heartsuit$. Let

$$0 \rightarrowtail d_1 \rightarrowtail d_2 \rightarrowtail \cdots \rightarrowtail d_k \rightarrowtail d$$

be the a filtration on d with associated graded in $\mathcal{C}^\heartsuit$. Since $c/d_n \cong (c/d_{n-1})/(d_n/d_{n-1})$, and since by induction both the numerator and denominator on the right hand side are in $\mathcal{C}^\heartsuit$, we get by induction that c/d is in $\mathcal{C}^\heartsuit$ also, as required. □

Lemma 5.9.13 ([BL21, Prop. 3.9])**.** *Let $f\colon A \to B$ be a map of $\mathbb{E}_1$–rings such that:*

1. $\mathrm{Perf}(A)$ *is bounded and regular,*

2. $\mathrm{cofib}(f)$ *has Tor–amplitude in $[-\infty, -1]$ as an A–right module.*

Then $\mathrm{Perf}(B)$ *is bounded and regular, and the basechange functor* $- \otimes_A B\colon \mathrm{Perf}(A) \to \mathrm{Perf}(B)$ *is t–exact inducing an equivalence* $\mathrm{K}(A) \to \mathrm{K}(B)$*, and moreover,* $\mathbb{K}_{-1}(A) = \mathbb{K}_{-1}(B) = 0$.

Proof. Condition (2) on Tor–amplitudes guarantee that $\mathrm{cofib}(f) \otimes_A N \simeq \mathrm{cofib}(N \to F^*F_!N)$ is coconnected when $N \in \mathrm{LMod}(A)^\heartsuit$. Hence, the condition of Lemma 5.9.6 is satisfied, and since $B \otimes_A -\colon \mathrm{Perf}(A) \to \mathrm{Perf}(B)$ is clearly quasi–affine as A is sent to B, we can apply Theorem 5.9.12 to obtain the desired statement. □

Theorem 5.9.14 ([BL21, Thm. 1.1])**.** *Let R be a coconnective $\mathbb{E}_1$–algebra such that:*

1. $\pi_0 R$ *is left regular coherent,*

2. $\tau_{\leq -1}R$ *has Tor–amplitude in $[-\infty, -1]$ as a $\pi_0 R$–right module.*

Then the natural map $\mathrm{K}(\pi_0 R) \to \mathrm{K}(R)$ *is an equivalence and both* π_0 *and* R *have vanishing* $\mathbb{K}_{-1}$*.*

Proof. Condition (1) ensures, via Proposition 5.7.23, that the t–structure on $\mathrm{RMod}(\pi_0 R)$ restricts to a bounded one on $\mathrm{Perf}(\pi_0 R)$, and condition (2) ensures that we can apply Lemma 5.9.13 to obtain the desired conclusions. □

Using Theorem 5.9.14, we may obtain another proof (different from the standard one, which we did not cover in the lectures) of the Quillen localisation sequence.

Theorem 5.9.15 (Quillen's localisation sequence, [BL21, Ex. 1.2])**.** *There is a fibre sequence of connective spectra* $\mathrm{K}(\mathbb{F}_p) \longrightarrow \mathrm{K}(\mathbb{Z}) \longrightarrow \mathrm{K}(\mathbb{Z}[1/p])$.

Proof. Write $\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}} := (\mathrm{Mod}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}})^{\omega}$. Applying Theorem 4.5.46 to the Karoubi sequence $\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}} \hookrightarrow \mathrm{Perf}(\mathbb{Z}) \twoheadrightarrow \mathrm{Perf}(\mathbb{Z}[1/p])$, we obtain a fibre sequence of spectra

$$\mathbb{K}(\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}) \longrightarrow \mathbb{K}(\mathbb{Z}) \longrightarrow \mathbb{K}(\mathbb{Z}[1/p]).$$

Since all the categories involved are idempotent–complete, applying $\tau_{\geq 0}$ to these non-connective K–theory spectra yields a fibre sequence in $\mathrm{Sp}_{\geq 0} \simeq \mathrm{CGrp}$ of connective K–theories. Hence, by Exercise 4.4.7, to obtain the fibre sequence of connective spectra as in the statement, we need to argue that $\mathbb{K}_{-1}(\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}) = 0$ and that $\mathrm{K}(\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}) \simeq \mathrm{K}(\mathbb{F}_p)$. For this, first note that $\mathrm{Mod}(\mathbb{Z})^{p-\mathrm{nil}} \simeq \mathrm{Mod}(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p))$. Indeed, by Theorem 2.6.6, we only need to show that $\mathbb{F}_p \in \mathrm{Mod}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}$ is a compact generator. That it is compact is clear, and to see that it generates, suppose $\mathrm{map}_{\mathbb{Z}}(\mathbb{F}_p, X) \simeq 0$ for some $X \in \mathrm{Mod}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}$. Then the map $p\colon X \simeq \mathrm{map}_{\mathbb{Z}}(\mathbb{Z}, X) \longrightarrow \mathrm{map}_{\mathbb{Z}}(\mathbb{Z}, X) \simeq X$ is an equivalence. Thus, $X \in \mathrm{Mod}(\mathbb{Z}[1/p])$ also, and so $X \simeq 0$ as required. Having shown this, since $\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p) \simeq \mathbb{F}_p \oplus \mathbb{F}_p[-1]$, the conditions of Theorem 5.9.14 clearly hold and we can apply the theorem to get simultaneously that $\mathrm{K}(\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}) \simeq \mathrm{K}(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p)) \simeq \mathrm{K}(\mathbb{F}_p)$ and that $\mathbb{K}_{-1}(\mathrm{Perf}(\mathbb{Z})^{\mathbb{Z}[1/p]-\mathrm{tors}}) \cong \mathbb{K}_{-1}(\mathrm{End}_{\mathbb{Z}}(\mathbb{F}_p)) = 0$. □

6 Algebraic K–theory: Land–Tamme circle–dot excision

6.1 Overview

We have seen in §4.5 that nonconnective K–theory $\mathbb{K}$ satisfies excellent descent properties (i.e. it is a localising invariant valued in spectra). In doing actual calculations in K–theory, one is naturally led to ask the following:

Questions 6.1.1 (Milnor descent)**.** Suppose we are given a square of rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B', \end{tikzcd} \tag{6.1}$$

which is either a pullback or a pushout, when does it happen that the induced square of (nonconnective) K–theory spectra

$$\begin{tikzcd} \mathbb{K}(A) \arrow[r] \arrow[d] & \mathbb{K}(B) \arrow[d] \\ \mathbb{K}(A') \arrow[r] & \mathbb{K}(B') \end{tikzcd}$$

is a pullback in Sp?

Such a pullback of K–theories would yield a long exact sequence

$$\cdots \to \mathbb{K}_2(A') \oplus \mathbb{K}_2(B) \longrightarrow \mathbb{K}_2(B') \longrightarrow \mathbb{K}_1(A) \longrightarrow \cdots$$

of abelian groups, which is very powerful for computations, and the search for such long exact sequences has a distinguished history. However, as pointed out in the introduction to [LT19], Swan showed that there are no functors "K_2" from rings to abelian groups fitting into such a long exact sequence of abelian groups.

This descent question was the fundamental question which formed the basis of the celebrated twin papers [LT19; LT23] of Markus Land and Georg Tamme. Their key insight is that one *does* have a pullback of K–theories by universally replacing either A or B', as the following theorem makes more precise:

Theorem 6.1.2 (Theorem 6.5.10)**.** *Suppose we have a pullback square and a pushout square of* $\mathbb{E}_1$*–rings*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R' \arrow[r] & S' . \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Then there is are associated natural $\mathbb{E}_1$*–rings* $A' \odot_A^{B'} B$ *and* $R' \boxtimes_{S'} S$ *and refined commuting diagrams of* $\mathbb{E}_1$*–rings*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A' \arrow[r] \arrow[drr, bend right] & A' \odot_{B'} B \arrow[dr] & \\
& & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R' \boxtimes_{S'} S \arrow[r] \arrow[d] & S \arrow[d] \\
& R' \arrow[r] & S'
\end{tikzcd}$$

whose inner squares are sent to pullback squares by K–theory (in fact, by any localising invariant valued in a stable category). The underlying spectrum of $A' \odot_A^{B'} B$ *is equivalent to* $A' \otimes_A B$ *and the underlying spectrum of* $R' \boxtimes_{S'} S$ *is equivalent to the pullback* $R' \times_{S'} S$*. The underlying diagrams of spectra are the canonical ones.*

This theorem provides a very general strategy to answer the Question 6.1.1: since we already have pullback squares of spectra

$$\begin{tikzcd}
\mathbb{K}(A) \arrow[r] \arrow[d] & \mathbb{K}(B) \arrow[d] \\
\mathbb{K}(A') \arrow[r] & \mathbb{K}(A' \odot_{B'} B)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{K}(R' \boxtimes_{S'} S) \arrow[r] \arrow[d] & \mathbb{K}(S) \arrow[d] \\
\mathbb{K}(R') \arrow[r] & \mathbb{K}(S'),
\end{tikzcd}$$

the problem now is to show that $\mathbb{K}(A' \odot_{B'} B) \to \mathbb{K}(B')$ and $\mathbb{K}(R) \to \mathbb{K}(R' \boxtimes_{S'} S)$ are equivalences. In other words, this turns a question about pullbacks of rings into a question about pushouts of rings and vice versa. These are surprisingly tractable problems – essentially by virtue of the concrete descriptions of $A' \odot_{B'} B$ and $R' \boxtimes_{S'} S$ given in the theorem – that can be dealt with in many concrete situations, and indeed [LT19; LT23] go to great lengths in showing us how to approach such questions by unifying and generalising many famous old results with their theory.

The goal of this chapter is to state and prove the main theorems of these papers in the form of Theorems 6.5.1, 6.5.4, 6.5.9 and 6.5.10. The decisive construction of these papers is the *circle–dot–construction*, which in turn rests upon the notion of oriented pullbacks. The key magic of all these things turn out to be quite simple, as encapsulated by Proposition 6.5.2. We will treat all of these in detail in the coming sections before coming to the proofs of the main results. To end the chapter, we will see a selection of very cool applications as presented in [LT19; LT23].

6.2 The technology of oriented pullbacks

This is mostly based on the theory developed in [Tam18].

Construction 6.2.1 (Oriented pullbacks)**.** Given a cospan $\mathcal{A} \xrightarrow{p} \mathcal{C} \xleftarrow{q} \mathcal{B}$ of stable categories, we can define the *oriented pullback* as the pullback in the diagram

$$\begin{tikzcd}
\mathcal{A}\overrightarrow{\times}_{p,\mathcal{C},q}\mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d, "s\times t"] \\
\mathcal{A}\times\mathcal{B} \arrow[r, "p\times q"'] & \mathcal{C}\times\mathcal{C}
\end{tikzcd}$$

Note that there is an asymmetry in lax pullbacks that are not seen in ordinary pullbacks. As such, objects in $\mathcal{A}\overrightarrow{\times}_{p,\mathcal{C},q}\mathcal{B}$ look like triples (X, Y, f) where $X \in \mathcal{A}, Y \in \mathcal{B}$, and $f\colon pX \to qY$. If the maps p, q are clear, we will also often opt for the shorter notation $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$.

Note that an oriented pullback canonically fits into a lax square

$$\begin{tikzcd}
\mathcal{A}\overrightarrow{\times}_{p,\mathcal{C},q}\mathcal{B} \arrow[r, "\mathrm{pr}_{\mathcal{B}}"] \arrow[d, "\mathrm{pr}_{\mathcal{A}}"'] & \mathcal{B} \arrow[d, "q"] \\
\mathcal{A} \arrow[r, "p"'] \arrow[ur, Rightarrow, "f"] & \mathcal{C}
\end{tikzcd} \tag{6.2}$$

where the transformation f is given by the map $f\colon pX \to qY)$ associated to the object $(X, Y, pX \xrightarrow{f} qY)$.

Observation 6.2.2. Unwinding the formula for mapping anima in pullbacks, for $(A, B, pA \xrightarrow{f} qB), (A', B', pA' \xrightarrow{f'} qB') \in \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$, we have the formula

$$\begin{tikzcd}
\mathrm{Map}_{\overrightarrow{\times}}\big((A, B, pA \xrightarrow{f} qB), (A', B', pA' \xrightarrow{f'} qB')\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{B}}(B, B') \arrow[d, "f^*\circ q"] \\
\mathrm{Map}_{\mathcal{A}}(A, A') \arrow[r, "f'_*\circ p"'] & \mathrm{Map}_{\mathcal{C}}(pA, qB')
\end{tikzcd}$$

Using this, we see that we have fully faithful inclusions

$$j_1\colon \mathcal{A} \hookrightarrow \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B} \quad :: \quad A \mapsto (A, 0, pA \to 0)$$

$$j_2\colon \mathcal{B} \hookrightarrow \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B} \quad :: \quad B \mapsto (0, B, 0 \to qB)$$

Moreover, by [Lur12, Lem. 5.4.5.2, Lem. 5.4.5.4] colimits are formed pointwise. In particular, for every $(A, B, pA \xrightarrow{f} qB) \in \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$, we have the key fibre sequence

$$(0, B, 0 \to qB) \to (A, B, pA \xrightarrow{f} qB) \to (A, 0, pA \to 0)$$

Proposition 6.2.3 ([Tam18, Prop. 10], [LT19, Lem. 1.5])**.** *There is a right–split Verdier sequence*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, hook, "j_2"] & \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B} \arrow[r, two heads, "\pi_1"] \arrow[l, bend left, "\pi_2"] & \mathcal{A} \arrow[l, bend left, hook, "j_1"]
\end{tikzcd}$$

Moreover, if $p\colon \mathcal{A} \to \mathcal{C}$ had a right adjoint r, then $j_1\colon \mathcal{A} \to \mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$ admits also a right adjoint, which is concretely given by

$$(A, B, pA \xrightarrow{f} qB) \mapsto \operatorname{fib}(A \to rpA \xrightarrow{rf} uqB)$$

Proof. For the first part, by the usual yoga of Verdier sequences, it is enough to show one of the adjoints, and this is not hard using the mapping anima formula above. To see the second part, let $A' \in \mathcal{A}$. Then by the formula in Observation 6.2.2 we have

$$\begin{tikzcd}
\operatorname{Map}_{\overrightarrow{\times}}\big((A', 0, pA' \to 0), (A, B, pA \xrightarrow{f} qB)\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Map}_{\mathcal{B}}(0, B) \simeq 0 \arrow[d] \\
\operatorname{Map}_{\mathcal{A}}(A', A) \arrow[r, "p"'] & \operatorname{Map}_{\mathcal{C}}(pA', qB) \simeq \operatorname{Map}_{\mathcal{A}}(A', rqB)
\end{tikzcd}$$

whence the desired conclusion. □

Lemma 6.2.4 ([LT19, Lem. 1.9])**.** *If $\mathcal{A}$ and $\mathcal{B}$ are generated by a sets of objects S and T respectively, then $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$ is generated by the set $\{(X, 0, 0) | X \in S\} \cup \{(0, Y, 0) | Y \in T\}$.*

Proof. This is because we have fibre sequences $(0, N, 0) \to (M, N, f) \to (M, 0, 0)$. □

Lemma 6.2.5 ([LT23, Lem. 2.2])**.** *Suppose that the functor $p\colon \mathcal{A} \to \mathcal{C}$ admits a right adjoint $r\colon \mathcal{C} \to \mathcal{A}$. Then there is a canonical equivalence*

$$\mathcal{A}\overrightarrow{\times}_{p,\mathcal{C},q}\mathcal{B} \xrightarrow{\simeq} \mathcal{A}\overrightarrow{\times}_{\mathrm{id},\mathcal{A},rq}\mathcal{B} \quad :: \quad (X, Y, pX \xrightarrow{f} qY) \mapsto (X, Y, X \xrightarrow{\widehat{f}} rqY)$$

where $\widehat{f}$ is the adjoint of f. Similarly, if q admist a left adjoint $s\colon \mathcal{C} \to \mathcal{B}$, then there is a canonical equivalence

$$\mathcal{A}\overrightarrow{\times}_{p,\mathcal{C},q}\mathcal{B} \xrightarrow{\simeq} \mathcal{A}\overrightarrow{\times}_{sp,\mathcal{B},\mathrm{id}}\mathcal{B} \quad :: \quad (X, Y, pX \xrightarrow{f} qY) \mapsto (X, Y, spX \xrightarrow{\widehat{f}} Y)$$

where again $\widehat{f}$ is the adjoint of f.

Lemma 6.2.6 ([LT23, Lem. 2.3])**.** *Let $\mathcal{A} \xrightarrow{p} \mathcal{C} \xleftarrow{q} \mathcal{B}$ be a diagram of stable categories and assume that $\mathcal{A}, \mathcal{B}$ are compactly generated presentable, and $\mathcal{C}$ is cocomplete, and that p, q preserve small colimits, and moreover that p preserves compact objects. Then $\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}$ is compactly generated presentable stable and $(\mathcal{A}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B})^{\omega} \simeq \mathcal{A}^{\omega}\overrightarrow{\times}_{\mathcal{C}}\mathcal{B}^{\omega}$.*

6.3 The circle–dot–construction for rings

Throughout, k will be a base $\mathbb{E}_2$–ring spectrum. We write $\mathrm{Cat}^k := \mathrm{Mod}_{\mathrm{Perf}(k)}(\mathrm{Cat}^{\mathrm{perf}})$ for the category of small, idempotent–complete, k–linear categories.

The construction

Definition 6.3.1 (Milnor contexts, [LT23, §2.2])**.** A *Milnor context (over k)* consists of a triple (A', B, M) where A' and B are objects in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(k))$ and M is a pointed (B, A')–bimodule in $\mathrm{Mod}(k)$, i.e. a (B, A')–bimodule equipped with maps a k–linear map $k \to M$ and maps $A' \xrightarrow{m} M \xleftarrow{m} B$ where the first map is right A'–linear and the second is left B–linear. Equivalently, it is a (B, A')–bimodule M equipped with a (B, A')–bimodule map $B \otimes_k A' \to M$.

Construction 6.3.2 ($A' \boxtimes_M B$, [LT23, Cons. 2.5])**.** By Morita theory (c.f. [Lur17, Rmk. 4.8.4.9]), the datum of a bimodule M is equivalent to the datum of a colimit–preserving and k–linear functor $M := (-) \otimes_B M \colon \mathrm{RMod}(B) \to \mathrm{RMod}(A')$. We define

$$\mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) := \mathrm{RMod}(A')\overrightarrow{\times}_{\mathrm{id},\mathrm{RMod}(A'),M}\mathrm{RMod}(B)$$

so that an object consists of a triple (X, Y, f) where $X \in \mathrm{RMod}(A')$, $Y \in \mathrm{RMod}(B)$, and $f \colon X \to Y \otimes_B M$ is a map of right A'–modules.

We have yet to use the basepoint structures so far, and we utilise them now. We define $\Lambda := (A', B, A' \xrightarrow{m} M) \in \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$ and

$$A := A' \boxtimes_M B := \mathrm{End}_{\overrightarrow{\times}}(\Lambda). \tag{6.3}$$

By Lemma 6.2.6, we see that Λ is a compact object. Thus, by Theorem 2.6.6, we obtain a fully faithful colimit–preserving functor

$$i \colon \mathrm{RMod}(A) \longrightarrow \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) \tag{6.4}$$

sending A to Λ.

Lemma 6.3.3 ([LT23, Lem. 2.7])**.** *There is a canonical pullback diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "m"] \\ A' \arrow[r, "m"] & M \end{tikzcd}$$

of (A, A)–bimodules, where the maps $A \to A'$ and $B \to B$ are ring maps.

Proof. By Observation 6.2.2, we have the following pullback of spectra

$$\begin{tikzcd} \mathrm{map}_{\overrightarrow{\times}}(\Lambda, \Lambda) \arrow[r] \arrow[d] & \mathrm{map}_B(B, B) \arrow[d] \\ \mathrm{map}_{A'}(A', A') \arrow[r, "m_*"] & \mathrm{map}_{A'}(A', M). \end{tikzcd}$$

It is then straightforward to see that this pullback square identifies with the square in the statement. ☐

Observation 6.3.4. Note that the compositions

$$i_1\colon \mathrm{RMod}(A) \overset{i}{\hookrightarrow} \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) \xrightarrow{\text{proj}} \mathrm{RMod}(A')$$

$$i_2\colon \mathrm{RMod}(A) \overset{i}{\hookrightarrow} \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) \xrightarrow{\text{proj}} \mathrm{RMod}(B)$$

are effected by basechanging along the ring maps from Lemma 6.3.3.

Construction 6.3.5 (Circle–dot category and ring, [LT23, Cons. 2.8, Prop. 2.9])**.** We define $\mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$ to be the Verdier quotient in the Verdier sequence

$$\mathrm{RMod}(A) \xhookrightarrow{\ i\ } \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) \xrightarrow{\ p\ } \mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$$

using the inclusion map i from (6.4). We then define $A' \odot_M B$ to be the k–algebra

$$A' \odot_M B := \mathrm{End}_{\mathrm{RMod}(A')\odot_M\mathrm{RMod}(B)}(p(0,B,0)).$$

Note that since $(0,B,0) \in \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$ was compact by Lemma 6.2.6, we see that $p(0,B,0) \in \mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$ is compact too.

Proposition 6.3.6 ([LT23, Prop. 2.9])**.** *As a cocomplete k–linear category, $\mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$ is generated by $p(0,B,0)$. In particular, $\mathrm{RMod}(A') \odot_M \mathrm{RMod}(B) \simeq \mathrm{RMod}(A' \odot_M B)$.*

Proof. By Lemma 6.2.4, it is immediate that $(0,B,0)$ and $(A',0,0)$ is a set of compact generators for $\mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$. Furthermore, observe that (A',B,m) is in the image of the inclusion (6.4) by construction of A. Now, observe that we have a fibre sequence in $\mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$

$$(0,B,0) \longrightarrow (A',B,m) \longrightarrow (A',0,0)$$

and so since $p(A',B,m) \simeq 0 \in \mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$, we see that $p(0,B,0) \simeq \Omega p(A',0,0)$. Hence, $p(0,B,0)$ is a compact generator in $\mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)$ as claimed. The last statement is just an application of Theorem 2.6.6. □

Construction 6.3.7. Recall the maps i_1, i_2 from Observation 6.3.4. We write also

$$j_1\colon \mathrm{RMod}(A') \to \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$$

$$j_2\colon \mathrm{RMod}(B) \to \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B)$$

for the inclusion functors from Observation 6.2.2. Now consider the diagram

$$\begin{tikzcd}
\mathrm{RMod}(A) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{RMod}(B) \arrow[d, hook, "j_2"] \arrow[rdd, bend left, "pj_2"] & \\
\mathrm{RMod}(A') \arrow[r, hook, "\Omega j_1"'] \arrow[ru, Rightarrow, "\tau"] \arrow[rrd, bend right, "\Omega pj_1"'] & \mathrm{RMod}(A')\overrightarrow{\times}_M\mathrm{RMod}(B) \arrow[rd, two heads, "p"] & \\
 & & \mathrm{RMod}(A') \odot_M \mathrm{RMod}(B)
\end{tikzcd}$$

where k_1 and k_2 are just defined to be the appropriate composites, and where $\tau\colon \Omega j_1 \circ i_1 \Rightarrow j_2 \circ i_2$ comes from the rotated sequence of Observation 6.2.2

$$(\Omega X \otimes_A A', 0, 0) \xrightarrow{\tau} (0, X \otimes_A B, 0) \longrightarrow (X \otimes_A A', X \otimes_A B, m)$$

where m is the map $\mathrm{id} \otimes m\colon X \otimes_A A' \to X \otimes_A M \simeq (X \otimes_A B) \otimes_B M$. As we saw in Construction 6.3.5, the functor p sends τ to an equivalence, and so the outer square commutes.

Example 6.3.8 (Pullback of rings). Here is the key example that formed the basis of [LT19]. Suppose we have a pullback of $\mathbb{E}_1$–rings

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd} \tag{6.5}$$

giving rise to a Milnor context (A', B, B') whose associated ring $A' \boxtimes_{B'} B$ agrees with the pullback ring A given in (6.5) by virtue of Lemma 6.3.3.

There is then a functor $c\colon \mathrm{Perf}(A')\overrightarrow{\times}_{\mathrm{Perf}(B')}\mathrm{Perf}(B) \to \mathrm{Perf}(B')$ given by $(M, N, M \xrightarrow{f} N \otimes_B B') \mapsto \mathrm{cofib}(M \otimes_A B' \xrightarrow{f} N \otimes_B B')$. This moreover fits into a diagram

$$\begin{tikzcd}
\mathrm{Perf}(A) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{Perf}(B) \arrow[d, hook, "j_2"] \arrow[ddr, bend left, "k_2"] \arrow[ddr, phantom, "\Rightarrow"] & \\
\mathrm{Perf}(A') \arrow[ur, Rightarrow, "\tau"] \arrow[r, hook, "\Omega j_1"'] \arrow[drr, bend right, "k_1"'] & \mathrm{Perf}(A')\overrightarrow{\times}_{\mathrm{Perf}(B')}\mathrm{Perf}(B) \arrow[dr, "c"] \arrow[d, phantom, "\Downarrow"] & \\
& & \mathrm{Perf}(B')
\end{tikzcd}$$

where k_1 and k_2 are the maps induced by the ring maps $A' \to B'$ and $B \to B'$, and where the lax/commutation structures are given by:

- $\tau\colon \Omega j_1 \circ i_1 \Rightarrow j_2 \circ i_2$ comes from the rotated sequence of Observation 6.2.2

  $$(\Omega M \otimes_A A', 0, 0) \xrightarrow{\tau} (0, M \otimes_A B, 0) \longrightarrow (M \otimes_A A', M \otimes_A B, \mathrm{can})$$

  where can is the equivalence $(M \otimes_A A') \otimes'_A B' \simeq (M \otimes_A B) \otimes_B B'$,

- $\alpha_1\colon c \circ \Omega j_1 \xrightarrow{\simeq} k_1$ is given by

  $$\alpha_1\colon c(\Omega M, 0, 0) \simeq \mathrm{cofib}(\Omega M \otimes'_A B' \to 0) \simeq M \otimes_{A'} B' \xrightarrow{\simeq} M \otimes_{A'} B'$$

- $\alpha_2\colon c \circ j_2 \xrightarrow{\simeq} k_2$ is given by

  $$\alpha_2\colon c(0, N, 0) \simeq \mathrm{cofib}(0 \to N \otimes_B B') \xrightarrow{\simeq} N \otimes_B B'$$

Observe that these natural transformations paste to give the datum of a commuting square equivalent to the canonical one obtained by applying $\mathrm{Perf}(-)$ to (6.5). This is tantamount to checking that the diagram of natural transformations

$$\begin{tikzcd}
c\Omega j_1 i_1(M) \arrow[r, "\alpha_1", "\simeq"'] \arrow[d, "c\tau_M"'] & k_1 i_1(M) \simeq (M \otimes_A A') \otimes_{A'} B' \arrow[d, "\simeq"', "\mathrm{can}"] \\
c j_2 i_2(X) \arrow[r, "\alpha_2", "\simeq"'] & k_2 i_2(X) \simeq (M \otimes_A B) \otimes_B B'
\end{tikzcd}$$

commutes naturally in M. To see this, consider the defining cofibre sequence

$$(\Omega M \otimes_A A', 0, 0) \xrightarrow{\tau_M} (0, M \otimes_A B, 0) \longrightarrow (M \otimes_A A', M \otimes_A B, \mathrm{can})$$

Applying the cofibre construction in the definition of c on this cofibre sequence, we obtain the required identification of $c\tau_M$ with can by virtue of the following diagram of cofibre sequences

$$\begin{tikzcd}
\Omega(M \otimes_A A') \otimes_{A'} B' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (M \otimes_A A') \otimes_{A'} B' \arrow[d, "\mathrm{can}"] \\
0 \arrow[r] \arrow[d] & (\mathcal{M} \otimes_A B) \otimes_B B' \arrow[r, equal] \arrow[d, equal] & (M \otimes_A B) \otimes_B B' \arrow[d] \\
(M \otimes_A A') \otimes_{A'} B' \arrow[r, "c\tau_M"] & (M \otimes_A B) \otimes_B B' \arrow[r] & Q
\end{tikzcd}$$

where Q is both $\mathrm{cofib}(c\tau_M)$ and $\mathrm{cofib}(\mathrm{can})$.

Moreover, note that $c\colon \mathrm{Perf}(A')\overrightarrow{\times}_{\mathrm{Perf}(B')}\mathrm{Perf}(B) \to \mathrm{Perf}(B')$ vanishes on the full subcategory $\mathrm{Perf}(A)$. Hence, we obtain a unique factorisation of c through $\bar{c}\colon \mathrm{RMod}(A' \odot_M B) \to \mathrm{RMod}(B')$, so that as in Construction 6.3.7, we obtain a commuting diagram

$$\begin{tikzcd}
\mathrm{RMod}(A) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{RMod}(B) \arrow[d, hook, "pj_2"] \arrow[ddr, bend left, "k_2"] & \\
\mathrm{RMod}(A') \arrow[r, hook, "\Omega pj_1"'] \arrow[drr, bend right, "k_1"] & \mathrm{RMod}(A' \odot_M B) \arrow[dr, "\bar{c}"] & \\
 & & \mathrm{RMod}(B')
\end{tikzcd} \tag{6.6}$$

where, by virtue of the previous paragraph, the outer diagram is the one induced by (6.5).

In order to analyse this construction, it would be beneficial to record the following highly structure version of Schwede–Shipley. For this, recall that $\mathrm{Alg}_{\mathbb{E}_0}$ is just an object in a symmetric monoidal category together with a map from the tensor unit. In particular, $\mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Cat}^{\mathrm{perf}})$ consists of pairs $(\mathcal{C}, C)$ where $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and $C \in \mathcal{C}$.

Lemma 6.3.9 (Functorial Lurie–Schwede–Shipley, [LT19, Lem. 1.10])**.** *The association* $R \mapsto (\mathrm{Perf}(R), R)$ *extends to a fully faithful functor* $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Cat}^{\mathrm{perf}})$ *whose essential image consists of pairs* $(\mathcal{C}, C)$ *for which* C *generates* $\mathcal{C}$*, and its inverse is given by* $(\mathcal{C}, C) \mapsto \mathrm{map}_{\mathcal{C}}(C, C)$*.*

Corollary 6.3.10 ([LT19, pg 11])**.** *We have a commuting diagram of categories*

$$\begin{tikzcd}
\mathrm{RMod}(A) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{RMod}(B) \arrow[d, "pj_2"] \\
\mathrm{RMod}(A') \arrow[r, "p\Omega j_1"] & \mathrm{RMod}(A' \odot_A^{B'} B)
\end{tikzcd}$$

extracting the endomorphism rings of which yields a commuting square of $\mathbb{E}_1$*-rings*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & A' \odot_A^{B'} B.
\end{tikzcd} \tag{6.7}$$

In the situation of a Milnor context arising from a pullback of rings as in Example 6.3.8, the square of rings (6.7) refines the commuting square (6.5), i.e. there is a commuting diagram of $\mathbb{E}_1$*–rings*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[rdd, bend left] & \\
A' \arrow[r] \arrow[rrd, bend right] & A' \odot_A^{B'} B \arrow[rd] & \\
 & & B'.
\end{tikzcd}$$

Proof. Immediate from Construction 6.3.7 and Lemma 6.3.9. The second part in the case of pullbacks of rings is obtained by applying Lemma 6.3.9 to (6.6). ☐

Identifying the circle–dot ring

In this subsection, we will identify the abstractly defined circle–dot ring.

Notation 6.3.11. Taking the appropriate fibres in Lemma 6.3.3, we have the diagram

$$\begin{tikzcd}
 & I \arrow[r, equal] \arrow[d] & I \arrow[d] \\
J \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & B \arrow[d, "m"] \\
J \arrow[r] & A' \arrow[r, "m"] & M
\end{tikzcd}$$

Then I is automatically a (B,A)–bimodule. Viewed as the fibre of $A \to A'$, I is the (A,A)–bimodule obtained by forgetting the (B,A)–bimodule structure to an (A,A)–bimodule. Since I and B have canonical (B,A)– and (A,B)–bimodule structures, $I \otimes_A B$ has a canonical (B,B)–bimodule structure. We can similarly define $J \coloneqq \mathrm{fib}(A' \to B')$ with all the attendant bimodule considerations.

Proposition 6.3.12 ([LT19, Prop. 1.13, 1.14])**.** *Let (A', B, M) be a Milnor context. Then:*

1. *The underlying k–module of $A' \odot_M B$ is $A' \otimes_A B$.*
2. *Under (1), we have a fibre sequence of (B,B)–bimodules*
$$I \otimes_A B \longrightarrow B \longrightarrow A' \odot_M B.$$
3. *Under (1), the ring map $A' \to A' \odot_M B$ induced by the functor $\Omega p j_1$ is the obvious map $A' \to A' \otimes_A B$. Moreover, the underlying (A',A')–bimodule of $A' \odot_M B$ sits in a cofibre sequence*
$$A' \otimes_A J \longrightarrow A' \longrightarrow A' \odot_M B.$$

Proof. The strategy is to first prove the cofibre sequence in (2) and then use it to deduce (1). To establish notations, recall that by construction that we have a right split Verdier sequence

$$\begin{tikzcd}
\mathrm{RMod}(A) \arrow[r, hook, "i"] & \mathrm{RMod}(A') \overrightarrow{\times}_M \mathrm{RMod}(B) \arrow[r, two heads, "p"] \arrow[l, bend left, "s"] & \mathrm{RMod}(A' \odot_M B) \arrow[l, bend left, "r"]
\end{tikzcd}$$

where by general Verdier sequence yoga, the right adjoint r sits in a cofibre sequence in $\mathrm{RMod}(A') \overrightarrow{\times}_M \mathrm{RMod}(B)$

$$is \Longrightarrow \mathrm{id} \Longrightarrow rp \tag{6.8}$$

For the fibre sequence in question, note that the $\mathbb{E}_1$–ring map (and hence also a (B,B)–bimodule map) $B \to A' \odot_M B$ can also be described as $\mathrm{map}_{\overrightarrow{\times}}(j_2B, j_2B) \to \mathrm{map}_{\odot}(pj_2B, pj_2B)$. Using the $p \dashv r$ adjunction and the sequence (6.8) we obtain a cofibre sequence of (B,B)–bimodules

$$\mathrm{map}_{\overrightarrow{\times}}(j_2B, isj_2B) \longrightarrow \mathrm{map}_{\overrightarrow{\times}}(j_2B, j_2B) \longrightarrow \mathrm{map}_{\overrightarrow{\times}}(j_2B, rpj_2B)$$

Now, $sj_2B = s(0,B,0) = 0 \times_{B \otimes_B M} B \simeq 0 \times_M B \simeq I$ as an A–right module. Hence, using the adjunction $j_2 \dashv \pi_2$ from Proposition 6.2.3, the first map in the preceding sequence becomes

$$\mathrm{map}_B(B, I \otimes_A B) \longrightarrow \mathrm{map}_B(B,B)$$

Now it is a general fact that if X is a (B,B)–bimodule, then $X \simeq \mathrm{map}_B(B,M)$ as a (B,B)–bimodule. Hence this map is identified, as (B,B)–bimodules, as $I \otimes_A B \to B$, yielding the claimed sequence $I \otimes_A B \to B \to A' \odot_A^{B'} B$ of (B,B)–bimodules.

Next, we prove point (3). The strategy is to first repeat the arguments of point (2) to get the desired (A', A')–bimodule cofibre sequence, and then carry out an extra step of showing that this identification is compatible with the identification already made in (1). Recall that we had the map $\big(\tau\colon \Omega j_1 i_1 A \to j_2 i_2 A\big) \simeq \big(\tau\colon \Omega(A',0,0) \to (0,B,0)\big)$ which becomes an equivalence upon applying p. Consider the following map of vertical cofibre sequences

$$\begin{tikzcd}
\mathrm{map}_{\vec{\times}}(\Omega j_1 A', rp\Omega j_1 A') & \mathrm{map}_{\vec{\times}}(j_2 B, rp\Omega j_1 A') \arrow[l, "\tau^*"', "\simeq"] \arrow[r, "\tau_*", "\simeq"'] & \mathrm{map}_{\vec{\times}}(j_2 B, rpj_2 B) \\
\mathrm{map}_{\vec{\times}}(\Omega j_1 A', \Omega j_1 A') \arrow[u] & \mathrm{map}_{\vec{\times}}(j_2 B, \Omega j_1 A') \arrow[l] \arrow[r] \arrow[u] & \mathrm{map}_{\vec{\times}}(j_2 B, j_2 B) \arrow[u] \\
\mathrm{map}_{\vec{\times}}(\Omega j_1 A', is\Omega j_1 A') \arrow[u] & \mathrm{map}_{\vec{\times}}(j_2 B, is\Omega j_1 A') \arrow[l] \arrow[r] \arrow[u] & \mathrm{map}_{\vec{\times}}(j_2 B, isj_2 B) \arrow[u]
\end{tikzcd}$$

The inclusion $j_1\colon \mathrm{RMod}(A') \hookrightarrow \mathrm{RMod}(A')\vec{\times}_{\mathrm{RMod}(B')}\mathrm{RMod}(B)$ induces an $\mathbb{E}_1$–identification

$$A' \simeq \mathrm{map}_{A'}(\Omega A', \Omega A') \simeq \mathrm{map}_{\vec{\times}}(\Omega j_1 A', \Omega j_1 A')$$

and so the $\mathbb{E}_1$–map $A' \to A' \odot_A^{B'} B$ induced by $\Omega p j_1$ is equivalent to the top left vertical map. By the same methods as in the proof of point (2) and now using also the concrete description of the adjunction $j_1 \dashv r$ from Proposition 6.2.3, analysing the left vertical cofibre sequence gives the desired cofibre sequence in point (3). Now, the identification of point (1) expresses the top right term as $A' \otimes_A B$ and our task is to show that, together with the new identification of the top left term as $A' \otimes_A B$, the top horizontal composite is the identity. For this, note that the bottom two pullback squares can be identified as

$$\begin{tikzcd}
A' & 0 \arrow[l] \arrow[r] & B \\
A' \otimes_A J \arrow[u] & \Omega A' \otimes_A B \arrow[l] \arrow[r] \arrow[u] & I \otimes_A B \arrow[u]
\end{tikzcd}$$

and hence taking vertical cofibres, it becomes the identity on $A' \otimes_A B$. □

Proposition 6.3.13 ([LT19, Prop. 1.15])**.** *In the situation of a Milnor context arising from a pullback of rings as in Example 6.3.8. Under Proposition 6.3.12 (1), the ring map $A' \odot_A^{B'} B \to B'$ from Corollary 6.3.10 is given by the obvious map $A' \otimes_A B \to B'$ induced by the ring maps $A' \to B'$, $B \to B'$, and the multiplication on B'.*

Proof. By Corollary 6.3.10, the composite $\mathbb{E}_1$–maps $A' \to A' \odot_A^{B'} B \to B'$ and $B \to A' \odot_A^{B'} B \to B'$ are the given ring maps. In particular, the first composite implies that $A' \odot_A^{B'} B \to B'$ is A'–left linear. But then by point Proposition 6.3.12 (3), as an A'–left module, $A' \odot_A^{B'} B$ is equivalent to $A' \otimes_A B$, and so $A' \odot_A^{B'} B \to B'$ is the unique A'–left linear extension of the obvious map $B \to B'$, as required. □

Observation 6.3.14 ([LT19, Rmk. 1.16]). Suppose we have a pullback square of $\mathbb{E}_1$–rings as in (6.5). Write $I \coloneqq \mathrm{fib}(B \to B')$. By Proposition 6.3.13, the map $A' \odot_A^{B'} B \simeq A' \otimes_A B \to B'$ is also the unique B–right linear extension of the ring map $A' \to B'$. Hence, together with Proposition 6.3.12 (2), we obtain a map of cofibre sequences

$$\begin{tikzcd}
I \otimes_A B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & A' \odot_A^{B'} B \arrow[d] \\
I \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

where the left vertical map is the B–right module structure on I.

Proposition 6.3.15 ([LT23, Prop. 2.10]). *Let (A', B, M) be a Milnor context. Then the k–algebra map $A \to A' \odot_M B$ extends uniquely to an (A', B)–bimodule map $A' \otimes_A B \to A' \odot_M B$ which is an equivalence. Under this equivalence, the ring maps $A' \to A' \odot_M B$ and $B \to A' \odot_M B$ correspond to the canonical maps $A' \to A' \otimes_A B$ and $B \to A' \otimes_A B$, respectively.*

Proof. Firstly, note that via the ring maps from Corollary 6.3.10, $A' \odot_M B$ is canonically an (A, A)–bimodule and the map $A \to A' \odot_M B$ is an (A, A)–bimodule map. Since $A' \otimes_A B$ is the free (A', B)–bimodule on the (A, A)–bimodule A, there is a unique (A', B)–linear extension $A' \otimes_A B \to A' \odot_M B$ as claimed. The last statement in the proposition is then clear from this construction.

To see that $A' \otimes_A B \to A' \odot_M B$ is an equivalence as (A', B)–bimodules, consider the fibre sequence of (B, B)–bimodules

$$I \otimes_A B \longrightarrow B \longrightarrow A' \odot_M B$$

from Proposition 6.3.12 (2). On the other hand, we also have a fibre sequence of (A, A)–bimodules $I \longrightarrow A \longrightarrow A'$ and so by extension of scalars, a fibre sequence of (A, B)–bimodules

$$I \otimes_A B \longrightarrow B \longrightarrow A' \otimes_A B.$$

Thus, comparing with the previous fibre sequence, we obtain a canonical equivalence $A' \otimes_A B \xrightarrow{\simeq} A' \odot_M B$ as (A, B)–bimodules. To see that this map is (A', B)–linear, since we know already that it is (A, B)–linear, this map is the unique (A, B)–linear extension of the (A, A)–linear composite $A' \xrightarrow{\mathrm{id} \otimes B} A' \otimes_A B \xrightarrow{\simeq} A' \odot_M B$. By Proposition 6.3.12 (3), this composite *is* the k–algebra map $A' \to A' \odot_M B$ constructed above. In particular, it is (A', A)–linear. To finish, it is just now just a general fact that extending an (A', A)–linear map as an (A, A)–linear map to an (A, B)–linear map yields an (A', B)–linear map. This completes the proof. □

6.4 The circle–dot construction for categories

In this section, we use the circle–dot–construction to provide a candidate category that we will later prove in Theorem 6.5.1 to compute the pushout in Cat^k. This illustrates the second instance of the usefulness of the circle–dot philosophy.

Notation 6.4.1. Let k be an $\mathbb{E}_2$–ring. We write $\mathrm{Pr}^L_\omega(k)$ for the nonfull subcategory of $\mathrm{LMod}_{\mathrm{Pr}^L}(\mathrm{Mod}(k))$ on the compactly generated categories and compact–preserving functors. Note that since $\mathrm{Mod}(k)$ is stable and Sp is an idempotent algebra in Pr^L, being a module over $\mathrm{Mod}(k)$ automatically implies being a stable category.

Proposition 6.4.2 ([HSS17, Prop. 4.5 and 4.9])**.** *Let k be an $\mathbb{E}_2$–ring. The equivalence* $\mathrm{Ind}\colon \mathrm{Cat}^{\mathrm{perf}} \rightleftharpoons \mathrm{Pr}^L_{\mathrm{st},\omega} : (-)^\omega$ *induces an equivalence* $\mathrm{Ind}\colon \mathrm{Cat}^k \rightleftharpoons \mathrm{Pr}^L_\omega(k) : (-)^\omega$.

Construction 6.4.3 (Pushouts in Cat^k, [LT23, §3.1])**.** Consider a span $\mathcal{A}' \xleftarrow{u} \mathcal{A}_0 \xrightarrow{v} \mathcal{B}$ in Cat^k. Using the functor v^* from the adjunction $v_!\colon \mathrm{Ind}(\mathcal{A}_0) \rightleftharpoons \mathrm{Ind}(\mathcal{B}) : v^*$, we may form the oriented pullback

$$\mathcal{A}'\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathcal{B} \coloneqq \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},\mathrm{Ind}(\mathcal{A}'),u_!v^*}\mathcal{B} \tag{6.9}$$

whose objects consists of triples $(X, Y, X \to uv^*Y)$ with $X \in \mathcal{A}'$ and $Y \in \mathcal{B}$. Consider the canonical functor

$$i_0\colon \mathcal{A}_0 \longrightarrow \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},uv^*}\mathcal{B} \quad :: \quad Z \mapsto (u(Z), v(Z), u(Z \to v^*v_!(Z))) \tag{6.10}$$

and let $\mathcal{A} \subseteq \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},uv^*}\mathcal{B}$ denote the thick subcategory generated by the image of this functor. Furthermore, we define $\mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}$ to be the Verdier quotient fitting in the Verdier sequence

$$\mathcal{A} \overset{i}{\hookrightarrow} \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathcal{B} \overset{p}{\twoheadrightarrow} \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B} \coloneqq (\mathcal{A}'\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathcal{B}). \tag{6.11}$$

Writing

$$j_1\colon \mathcal{A}' \longrightarrow \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},uv^*}\mathcal{B} \qquad j_2\colon \mathcal{B} \longrightarrow \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},uv^*}\mathcal{B}$$

for the inclusion functors and

$$i_1\colon \mathcal{A} \hookrightarrow \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathcal{B} \xrightarrow{\mathrm{pr}_{\mathcal{A}'}} \mathcal{A}' \qquad i_2\colon \mathcal{A} \hookrightarrow \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathcal{B} \xrightarrow{\mathrm{pr}_{\mathcal{B}}} \mathcal{B}$$

for the compositions, we obtain as in Construction 6.3.7 a diagram

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[dr] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & & \\
& \mathcal{A} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathcal{B} \arrow[d, hook, "j_2"] \arrow[ddr, "pj_2"] & \\
& \mathcal{A}' \arrow[ur, Rightarrow, "\tau"] \arrow[r, "\Omega j_1"'] \arrow[drr, bend right, "\Omega p j_1"'] & \mathcal{A}'\overrightarrow{\times}_{\mathrm{id},uv^*}\mathcal{B} \arrow[dr, "p"] & \\
& & & \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}
\end{tikzcd} \tag{6.12}$$

where τ is induced by the natural fibre sequence

$$\Omega(X, 0, 0) \longrightarrow (0, Y, 0) \longrightarrow (X, Y, f).$$

By construction, $p(\tau)$ is a natural equivalence between functors $\mathcal{A} \to \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}$, we so we obtain commuting diagrams

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "v"] \arrow[d, "u"'] & \mathcal{B} \arrow[d, "pj_2"] & & \mathcal{A} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathcal{B} \arrow[d, "pj_2"] \\
\mathcal{A}' \arrow[r, "\Omega pj_1"'] & \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B} & & \mathcal{A}' \arrow[r, "\Omega pj_1"'] & \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}.
\end{tikzcd} \tag{6.13}$$

Example 6.4.4 ([LT23, Ex. 3.1]). If the span $\mathcal{A}' \leftarrow \mathcal{A}_0 \to \mathcal{B}$ is obtained from a span $A' \leftarrow A_0 \to B$ of k–algebras by applying $\mathrm{Perf}(-)$, then the oriented pullback (6.9) in this case is the oriented pullback associated to the Milnor context $(A', B, B \otimes_{A_0} A')$. Moreover, $\mathcal{A}$ in this case, as defined in Construction 6.4.3, identifies with $\mathrm{Perf}(A)$ where A is as in (6.3), since by a straightforward unwinding of definitions, $i_0(A_0)$ from (6.10) is given precisely by Λ from Construction 6.3.2.

Here is a string of lemmas which are instructive and not overly difficult to work out yourself. The proof of Lemma 6.4.5 also utilises the formula for the right adjoint of j_1 given in Proposition 6.2.3.

Lemma 6.4.5 ([LT23, Lem. 3.3])**.** *Suppose we have a span* $\mathcal{A}' \xleftarrow{u} \mathcal{A}_0 \xrightarrow{v} \mathcal{B}$ *in* Cat^k*. Then there is a canonical equivalence*

$$\Phi\colon \mathrm{Ind}(B)\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}') \xrightarrow{\simeq} \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B})$$

given by

$$(Y, X, v^*Y \xrightarrow{f} u^*X) \mapsto (\mathrm{fib}(\widehat{f}), Y, \mathrm{fib}(\widehat{f}) \to u_!v^*Y).$$

Moreover, this fits into a commuting diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}') \arrow[d, equal] & \mathrm{Ind}(B)\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}') \arrow[l, "\mathrm{pr}_{\mathcal{A}'}"'] \arrow[r, "\mathrm{pr}_{\mathcal{B}}"] \arrow[d, "\Phi", "\simeq"'] & \mathrm{Ind}(\mathcal{B}) \arrow[d, equal] \\
\mathrm{Ind}(\mathcal{A}') & \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B}) \arrow[l, "\Sigma j_{1*}"'] \arrow[r, "\mathrm{pr}_{\mathcal{B}}"] & \mathrm{Ind}(\mathcal{B})
\end{tikzcd}$$

where j_{1*} *denotes the right adjoint of* j_1*.*

Lemma 6.4.6 ([LT23, Lem. 3.5])**.** *The right adjoint of the functor* $i_{0!}\colon \mathrm{Ind}(\mathcal{A}_0) \to \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B})$ *is given by*

$$s_0\colon (X, Y, X \xrightarrow{f} u_!v^*Y) \mapsto u^*X \times_{u^*u_!v^*Y} v^*Y.$$

Construction 6.4.7 (Passing to right adjoints)**.** Given a pair of adjunctions (F, G, ϵ, η) and $(F', G', \epsilon', \eta')$ and a natural transformation $\tau\colon F \Rightarrow F'$, we may obtain an adjoint natural transformation given by

$$G' \xrightarrow{\eta_{G'}} GFG' \xrightarrow{G\tau_{G'}} GF'G' \xrightarrow{G\epsilon'} G.$$

Thus, by passing to the Ind–completions and right adjoints of all maps in (6.12) and using the identifications provided by Proposition 6.2.3 and Lemma 6.4.5, we get the lax square

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}_0) & \mathrm{Ind}(\mathcal{B}) \arrow[l, "v^*"'] \arrow[dl, Rightarrow, "\sigma'"] \\
\mathrm{Ind}(\mathcal{A}') \arrow[u, "u^*"] & \mathrm{Ind}(\mathcal{B})\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}') \arrow[l, "\mathrm{pr}_{\mathcal{A}'}"] \arrow[u, "\mathrm{pr}_{\mathcal{B}}"']
\end{tikzcd} \tag{6.14}$$

where the transformation σ' is given by

$$\sigma' \colon G'\Phi \xrightarrow{\eta_{G'}} GFG'\Phi \xrightarrow{G\tau_{G'}} GF'G'\Phi \xrightarrow{G\epsilon'} G\Phi \tag{6.15}$$

with

$$F = \Omega j_1 u_!, \quad F' = j_2 v_!, \quad G = u^*\Sigma j_{1*}, \quad G' = v^*\mathrm{pr}^2_{\mathcal{B}}.$$

Here, we have temporarily used the notations

$$\mathrm{pr}^1_{\mathcal{B}} \colon \mathrm{Ind}(\mathcal{B})\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}') \to \mathrm{Ind}(\mathcal{B}) \qquad \mathrm{pr}^2_{\mathcal{B}} \colon \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{B})$$

to disambiguate the two instances of projections.

With these preparations in place, we will state and prove the key lemma that will allow us to compute the pushout in Cat^k of $\mathcal{A}' \xleftarrow{u} \mathcal{A}_0 \xrightarrow{v} \mathcal{B}$ as $\mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}$.

Lemma 6.4.8 ([LT23, Lem. 3.4])**.** *The diagram (6.14) identifies with the canonical lax square (6.2) associated to the oriented pullback* $\mathcal{A}' \xrightarrow{u^*} \mathcal{A}_0 \xleftarrow{v^*} \mathcal{B}$*. In particular, restricted to the full subcategory of* $\mathrm{Ind}(\mathcal{B}) \times_{v^*,u^*} \mathrm{Ind}(\mathcal{A}') \subseteq \mathrm{Ind}(\mathcal{B})\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}')$ *on the objects* $(Y, X, v^*Y \xrightarrow{f} u^*X)$ *where f is an equivalence, the transformation σ' becomes an equivalence and so the resulting diagram is a pullback in* $\widehat{\mathrm{Cat}}$*.*

Proof. We analyse the transformation σ' from (6.15). By unwinding definitions, we obtain canonical identifications

$$\Sigma j_{1*} j_2 \simeq u_! v^*, \quad \Sigma j_{1*}\Phi \simeq \mathrm{pr}_{\mathcal{A}'}, \quad \mathrm{pr}^2_{\mathcal{B}}\Phi \simeq \mathrm{pr}^1_{\mathcal{B}} \tag{6.16}$$

which computes the transformation σ' from (6.15) as

$$v^*\mathrm{pr}^1_{\mathcal{B}} \xrightarrow{\eta} u^*u_!v^*\mathrm{pr}^1_{\mathcal{B}} \xrightarrow{u^*u_!\eta} u^*u_!(v^*v_!)v^*\mathrm{pr}^1_{\mathcal{B}} \longrightarrow u^*\mathrm{pr}_{\mathcal{A}'} \tag{6.17}$$

where the first two maps are identified with the appropriate adjunction units.

To understand the last map in (6.17), by (6.15), it is obtained by applying $u^*\Sigma j_{1*}(-)\Phi$ to the counit of $(F' \dashv G') = (j_2 v \dashv \mathrm{pr}^2_{\mathcal{B}} v^*)$, which factos as the counit of $(v_! \dashv v^*)$ followed by the counit of $(j_2 \dashv \mathrm{pr}^2_{\mathcal{B}})$. Therefore, using the canonical identifications (6.16), the final map in (6.17) is given by the composite

$$u^*u_!v^*(v_!v^*)\mathrm{pr}^2_{\mathcal{B}} \xrightarrow{u^*u_!v^*\epsilon} u^*u_!v^*\mathrm{pr}^1_{\mathcal{B}} \xrightarrow{\alpha} u^*\mathrm{pr}_{\mathcal{A}'}$$

where the map α is to be determined below. In any case, fitting this composite into (6.17), we obtain that the map of interest is given by

$$v^*\mathrm{pr}^1_{\mathcal{B}} \xrightarrow{\eta} u^*u_!v^*\mathrm{pr}^1_{\mathcal{B}} \xrightarrow{\alpha} u^*\mathrm{pr}_{\mathcal{A}'}. \tag{6.18}$$

To complete the proof, we analyse α. This is induced by the adjunction counit for $(j_2 \dashv \mathrm{pr}^2_{\mathcal{B}})$ on the functor Φ, or equivalently, the adjunction counit of $(\Phi^{-1}j_2 \dashv \mathrm{pr}^1_{\mathcal{B}} = \mathrm{pr}^2_{\mathcal{B}}\Phi)$. We first note that on an object $(F, Y, F \to u_!v^*Y)$ of $\mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B})$, the counit of the adjunction $(j_2 \dashv \mathrm{pr}^2_{\mathcal{B}})$ is the canonical map $(0, Y, 0) \to (F, Y, F \to u_!v^*Y)$. Thus, under the identification Φ from Lemma 6.4.5, the adjunction counit for $(\Phi^{-1}j_2 \dashv \mathrm{pr}^1_{\mathcal{B}})$ on an object $(Y, X, v^*Y \xrightarrow{f} u^*X)$ is the canonical map

$$(0, Y, 0) \longrightarrow (\mathrm{fib}(\widehat{f}), Y, \mathrm{fib}(\widehat{f}) \xrightarrow{\mathrm{can}} u_!v^*Y).$$

Now, applying Σj_{1*} to this natural transformation and using the formula of j_{1*} from Proposition 6.2.3, we obtain $u_!v^*Y \to \mathrm{cofib}(\mathrm{fib}(\widehat{f}) \xrightarrow{\mathrm{can}} u_!v^*Y) \simeq X$. Thus, applying u^* to this and precomposing with the unit of $(u_! \dashv u^*)$, we obtain the map f. That is, the map (6.18) applied to an object $(X, Y, v^*Y \xrightarrow{f} u^*X)$ is given by the map f. This completes the first part of the lemma, and the claim that we obtain a pullback square by restricting to the full subcategory of $\mathrm{Ind}(\mathcal{B})\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}')$ where f is an equivalence is now immediate. □

6.5 The main theorems of Land–Tamme

We can now combine the harvest of the previous sections to give the main theorems of [LT23] in the form of Theorems 6.5.1, 6.5.4, 6.5.9 and 6.5.10. We first present the abstract categorical pushout result.

Theorem 6.5.1 ([LT23, Thm. 3.2])**.** *The square*

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathcal{B} \arrow[d, "pj_2"] \\
\mathcal{A}' \arrow[r, "\Omega pj_1"'] & \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}
\end{tikzcd}$$

from Construction 6.4.3 is a pushout square in Cat^k*.*

Proof. By the equivalence $\mathrm{Cat}^k \simeq \mathrm{Pr}^L_\omega(k)$ from Proposition 6.4.2, we may equivalently show that the Ind–completed square

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}_0) \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathrm{Ind}(\mathcal{B}) \arrow[d, hook, "pj_2"] \\
\mathrm{Ind}(\mathcal{A}') \arrow[r, "\Omega pj_1"'] & \mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B})
\end{tikzcd}$$

is a pushout in $\mathrm{Pr}^L_\omega(k)$. We first claim that it suffices to show that this is a pushout in $\mathrm{Pr}^L_{\mathrm{st}}$. To this end, note that by construction, $\mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B})$ is a Verdier quotient of $\mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B})$, and by Lemma 6.2.6, $\mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B})$ is compactly generated by the images of the compact objects from $\mathrm{Ind}(\mathcal{A}')$ and $\mathrm{Ind}(\mathcal{B})$. Hence, if we are given a functor $\varphi\colon \mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B}) \to \mathcal{E}$ in $\mathrm{Pr}^L(k)$ such that both restrictions to $\mathrm{Ind}(\mathcal{A}')$ and $\mathrm{Ind}(\mathcal{B})$ preserve compact objects, then so does the functor φ. Therefore, it suffices to show that the square above is a pushout in $\mathrm{Pr}^L(k)$. But then, since $\mathrm{fgt}\colon \mathrm{Pr}^L(k) = \mathrm{LMod}_{\mathrm{Mod}(k)}(\mathrm{Pr}^L_{\mathrm{st}}) \to \mathrm{Pr}^L_{\mathrm{st}}$ is conservative and preserves colimits because (co)limits in module categories over presentably symmetric monoidal categories are computed underlying, it thus suffices to show that the square is a pushout in $\mathrm{Pr}^L_{\mathrm{st}}$ as claimed.

Now, by Observation 2.2.10, our problem is now equivalent to showing that the square of right adjoints

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}_0) & & \mathrm{Ind}(\mathcal{B}) \arrow[ll, "v^*"'] \\
\mathrm{Ind}(\mathcal{A}') \arrow[u, "u^*"] & \mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B}) \arrow[l, "\Sigma j_{1*}p_*"'] \arrow[ur, "\mathrm{pr}_{\mathcal{B}}p_*"']
\end{tikzcd}$$

is a pullback in $\widehat{\mathrm{Cat}}$, that is, we need to show that the canonical map $\mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{A}') \times_{u^*,v^*} \mathrm{Ind}(\mathcal{B})$ is an equivalence.

By construction, we have a right–split Verdier sequence

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{A}) \arrow[r, hook, "i"] & \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},uv^*}\mathrm{Ind}(\mathcal{B}) \arrow[l, bend left, "s"] \arrow[r, two heads, "p"] & \mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B}). \arrow[l, hook, bend left, "p_*"]
\end{tikzcd}$$

In particular, this identifies $\mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B})$ with $\ker(s)$. We write $s_0\colon \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},uv^*}\mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{A}_0)$ for the right adjoint of i_0. Now, by construction, the map $i_0\colon \mathrm{Ind}(\mathcal{A}_0) \to \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},uv^*}\mathrm{Ind}(\mathcal{B})$ factors through $i\colon \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},uv^*}\mathrm{Ind}(\mathcal{B})$ and $\mathrm{Ind}(\mathcal{A}_0) \to \mathrm{Ind}(\mathcal{A})$ generates the target under colimits, and so the right adjoint of $\mathrm{Ind}(\mathcal{A}_0) \to \mathrm{Ind}(\mathcal{A})$ is conservative and hence $\ker(s_0) = \ker(s) \simeq \mathrm{Ind}(\mathcal{A}') \odot_{\mathrm{Ind}(\mathcal{A}_0)} \mathrm{Ind}(\mathcal{B})$. We claim that the top equivalence Φ from Lemma 6.4.5 in

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{B})\overrightarrow{\times}_{v^*,u^*}\mathrm{Ind}(\mathcal{A}') \arrow[r, "\Phi", "\simeq"'] & \mathrm{Ind}(\mathcal{A}')\overrightarrow{\times}_{\mathrm{id},u_!v^*}\mathrm{Ind}(\mathcal{B}) \\
\mathrm{Ind}(\mathcal{B}) \times_{v^*,u^*} \mathrm{Ind}(\mathcal{A}') \arrow[u, hook] \arrow[r] & \ker(s_0) \arrow[u, hook]
\end{tikzcd} \tag{6.19}$$

restricts to the bottom equivalence. Here we have viewed, by virtue of Lemma 6.4.8, the pullback on the left hand side as the full subcategory of the oriented pullback consisting of those objects (X, Y, f) where f is an equivalence. By Lemma 6.4.8, the desired result will then be an immediate consequence of the claim.

To prove (6.19), note that obviously Φ restricts to an equivalence $\ker(s_0 \circ \Phi) \xrightarrow{\simeq} \ker(s_0)$, so we just need to determine $\ker(s_0 \circ \Phi)$. We thus have to show that

$(Y, X, v^*Y \xrightarrow{f} u^*X) \in \ker(s_0 \circ \Phi)$ if and only if f is an equivalence. This is now a straightforward application of the formula for Φ from Lemma 6.4.5 and the formula for s_0 from Lemma 6.4.6, since these formulas imply that we have the left pullback/pushout square in

$$\begin{tikzcd}
s_0\Phi \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & v^*Y \arrow[r] \arrow[d, "\eta"'] \arrow[dr, "f"] & \mathrm{cofib}(s_0\Phi \to v^*Y) \arrow[d, "\simeq"] \\
u^*\,\mathrm{fib}(\widehat{f}) \arrow[r] & u^*u_!v^*Y \arrow[r, "u^*\widehat{f}"] & u^*X
\end{tikzcd}$$

which implies that the right vertical is an equivalence. Hence, $s_0\Phi \simeq \mathrm{fib}(f)$, and so $s_0\Phi \simeq 0$ if and only if $\mathrm{fib}(f) \simeq 0$. ☐

Next, it would be good to isolate the standard motivic manoeuvre common to many of the results in the section.

Proposition 6.5.2 (Circle–dot descent, [LT19, After Lem. 1.11])**.** *Suppose we have a span* $\mathcal{A}' \xleftarrow{u} \mathcal{A}_0 \xrightarrow{v} \mathcal{B}$ *in* Cat^k. *Then the square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "i_2"] \arrow[d, "i_1"'] & \mathcal{B} \arrow[d, "pj_2"] \\
\mathcal{A}' \arrow[r, "\Omega pj_1"'] & \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}.
\end{tikzcd}$$

is a motivic pullback square.

Proof. Let $E\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a localising invariant. By the Verdier sequence defining $\mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}$ from (6.11), we get a fibre sequence in the stable category $\mathcal{E}$

$$E(\mathcal{A}) \longrightarrow E\Big(\mathcal{A}'\overrightarrow{\times}_{id,u_!v^*}\mathcal{B}\Big) \longrightarrow E(\mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}). \tag{6.20}$$

By Proposition 6.2.3, we obtain the equivalence in

$$E(i_1) \oplus E(i_2)\colon E(\mathcal{A}) \longrightarrow E\Big(\mathcal{A}'\overrightarrow{\times}_{id,u_!v^*}\mathcal{B}\Big) \xrightarrow[\simeq]{E(\mathrm{pr}_{\mathcal{A}'})\times E(\mathrm{pr}_{\mathcal{B}})} E(\mathcal{A}') \oplus E(\mathcal{B}).$$

Hence, the fibre sequence (6.20) shows that the square in the statement gets sent to a pullback by the functor E, as was to be shown. ☐

Corollary 6.5.3 ([LT23, Cor. 3.6])**.** *Suppose we have a pushout square in* Cat^k

$$\begin{tikzcd}
\mathcal{A}_0 \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathcal{B} \arrow[d] \\
\mathcal{A}' \arrow[r] & \mathcal{Q}.
\end{tikzcd}$$

Then with $\mathcal{A}$ *as constructed in Construction 6.4.3, the induced square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
\mathcal{A}' \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

is a motivic pullback square.

Proof. By Theorem 6.5.1, we know that $Q \simeq \mathcal{A}' \odot_{\mathcal{A}_0} \mathcal{B}$. Hence, the statement about the motivic pullback square is then obtained by Proposition 6.5.2. ☐

Theorem 6.5.4 ([LT23, Thm. 1.1])**.** *To a Milnor context (A', B, M), there are naturally associated k–algebras $A = A' \boxtimes_M B$ and $A' \odot_M B$, and a motivic pullback diagram*

$$\begin{tikzcd}
A' \boxtimes_M B \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & A' \odot_M B.
\end{tikzcd} \tag{6.21}$$

of k–algebras. Moreover, the underlying (A, A)–bimodule of $A' \boxtimes_M B$ canonically identifies with the pullback $A' \times_M B$, the underlying (A', B)–bimodule of $A' \odot_M B$ canonically identifies with $A' \otimes_A B$, and the underlying diagram of (A, A)–bimodules is the canonical one.

Proof. The last three statements are supplied by Lemma 6.3.3, Proposition 6.3.15, and Corollary 6.3.10. The statement about motivic pullback squares is by the arguments in Proposition 6.5.2. ☐

Definition 6.5.5 (Tensorisers)**.** Let (A', B, M) be a Milnor context and $A := A' \boxtimes_M B$ as in Construction 6.3.2. A *tensoriser for* (A', B, M) consists of k–algebra maps $A_0 \to A'$ and $A_0 \to B$ together with a commuting diagram

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & M
\end{tikzcd}$$

of (A_0, A_0)–bimodules, such that the induced (B, A')–bimodule map $B \otimes_{A_0} A' \to M$ is an equivalence. We will often abuse terminology and call A_0 the tensoriser.

In fact, tensorisers fit into the larger context of maps between Milnor contexts, which captures the functoriality of the circle–dot–constructions on rings.

Definition 6.5.6. A map of *Milnor contexts* $(C', D, N) \to (A', B, M)$ *over* k consists of k–algebra maps $C' \to A'$ and $D \to B$, a map of (D, C')–bimodules $N \to M$, and a commuting diagram

$$\begin{tikzcd}
N \arrow[rr] & & M \\
& k. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example 6.5.7. Using the notion of a map of Milnor contexts above, we may then phrase the datum of a tensoriser as that of a map of Milnor contexts $(A_0, A_0, A_0) \to (A', B, M)$ which has the further property that the (A_0, A_0)–bimodule map $A_0 \to M$ induces an equivalence $B \otimes_{A_0} A' \to M$ of (B, A')–bimodules.

Maps of Milnor contexts also allows us to articulate the functoriality of the constructions we have seen, which we will not prove in these notes and we refer the interested reader to the appropriate reference.

Lemma 6.5.8 ([LT23, Lem. 2.14])**.** *Given a map of Milnor contexts $(C', D, N) \to (B', A, M)$, we obtain a canonical map of commutative squares of k–algebras*

$$\begin{tikzcd}
C' \boxtimes_N D \arrow[r] \arrow[d] & D \arrow[d] & & A' \boxtimes_M B \arrow[r] \arrow[d] & B \arrow[d] \\
C' \arrow[r] & C' \odot_N D \arrow[r, phantom, "\longrightarrow"] & {} & A' \arrow[r] & A' \odot_M B.
\end{tikzcd}$$

Theorem 6.5.9 ([LT23, Thm. B])**.** *Let (A', B, M) be a Milnor context with tensoriser A_0. Then the canonical commuting square*

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & A' \odot_M B
\end{tikzcd}$$

from Theorem 6.5.4 and the map $A_0 \to A$ from Lemma 6.5.8 is a pushout diagram in $\mathrm{Alg}(k)$*.*

Proof. First, observe that the functor $\mathrm{Perf}(-)\colon \mathrm{Alg}(k) \to \mathrm{Cat}^k$ factors as the composite $\mathrm{Alg}(k) \to (\mathrm{Cat}^k)_{*/} \xrightarrow{\mathrm{fgt}} \mathrm{Cat}^k$ where the first functor sends A to the pair $(\mathrm{Perf}(A), A)$. As in Lemma 6.3.9, we know that the first functor is a fully faithful left adjoint with right adjoint given by sending $(\mathcal{C}, c)$ to $\mathrm{End}_{\mathcal{C}}(c)$. On the other hand, the second functor preserves contractible colimits by Exercise 2.1.22 and is conservative, and so in total the composite $\mathrm{Perf}(-)\colon \mathrm{Alg}(k) \to \mathrm{Cat}^k$ is conservative and preserves pushouts. All in all, it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{Perf}(A_0) \arrow[r] \arrow[d] & \mathrm{Perf}(B) \arrow[d] \\
\mathrm{Perf}(A') \arrow[r] & \mathrm{Perf}(A' \odot_M B)
\end{tikzcd}$$

is a pushout in Cat^k.

To this end, note that since A_0 was a tensoriser so that $B \otimes_{A_0} A' \xrightarrow{\simeq} M$ as (B, A')–bimodules, by Example 6.4.4, the pushout square in Theorem 6.5.1 is identified with the square

$$\begin{tikzcd}
\mathrm{Perf}(A_0) \arrow[r] \arrow[d] & \mathrm{Perf}(B) \arrow[d] \\
\mathrm{Perf}(A') \arrow[r] & \big(\mathrm{Perf}(A')\overrightarrow{\times}_M\mathrm{Perf}(B)\big)/\mathrm{Perf}(A) = \mathrm{Perf}(A') \odot_M \mathrm{Perf}(B).
\end{tikzcd}$$

But by Proposition 6.3.6, we have $\mathrm{Perf}(A') \odot_M \mathrm{Perf}(B) \simeq \mathrm{Perf}(A' \odot_M B)$. This thus completes the proof. ☐

Theorem 6.5.10 ([LT19, Main Thm.], [LT23, Thm. A])**.** *Suppose we have a pullback square and a pushout square of $\mathbb{E}_1$–k–algebras*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R' \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & S'.
\end{tikzcd}$$

Then there is are associated natural $\mathbb{E}_1$–k–algebras $A' \odot_A^{B'} B$ and $R' \boxtimes_{S'} S$ and refined commuting diagrams of k–algebras

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A' \arrow[r] \arrow[drr, bend right] & A' \odot_{B'} B \arrow[dr] & \\
& & B'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& R' \boxtimes_{S'} S \arrow[r] \arrow[d] & S \arrow[d] \\
& R' \arrow[r] & S'
\end{tikzcd}$$

whose inner squares are motivic pullback squares. The underlying k–module of $A' \odot_A^{B'} B$ is equivalent to $A' \otimes_A B$ and the underlying k–module of $R' \boxtimes_{S'} S$ is equivalent to the pullback $R' \times_{S'} S$. The underlying diagrams of k–modules are the canonical ones.

Proof. The first diagram is an immediate consequence of Theorem 6.5.4 and Corollary 6.3.10, and the second of Theorem 6.5.4 and Theorem 6.5.9, using the Milnor context $(R', S, S \otimes_R R')$ with tensoriser R. ☐

6.6 Application: quantitative K–theory descent for pullbacks

In this section, we explore various quantitative forms (in terms of connectivities) of descent afforded to us by the main theorems.

Definition 6.6.1. Suppose we have a commuting square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
Z \arrow[r] & W
\end{tikzcd}$$

of spectra. We say that it is *n–cartesian* if the map $X \to Y \times_W Z$ is n–connective (i.e. $\pi_r\,\mathrm{fib}(X \to Y \times_W Z) = 0$ for $r < n$).

Observation 6.6.2 (Interchanging connectivity questions, [LT19, Rmk. 2.2])*.* Suppose we have a commuting square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ Z \arrow[r, "f'"] & W \end{tikzcd}$$

in a stable category. Then by the equivalence (which we will prove below)

$$\Omega\,\mathrm{fib}(Z \cup_X Y \to W) \simeq \mathrm{fib}(X \to Z \times_W Y), \tag{6.22}$$

we deduce that the square above is n–cartesian if and only if the canonical map $Z \cup_X Y \to W$ is $(n+1)$–connective.

To see the equivalence, we note first that the total fibre $\mathrm{tofib} \coloneqq \mathrm{fib}(X \to Z \times_W Y)$ may be equivalently computed as the fibre in

$$\begin{tikzcd} \mathrm{tofib} \arrow[d] & & \\ \mathrm{fib}(f) \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ \mathrm{fib}(f') \arrow[r] & Z \arrow[r, "f'"] & W. \end{tikzcd}$$

This is because $\mathrm{tofib} \coloneqq X \times_{Y \times_W Z} 0$ is equivalent to

$$\simeq (X \times_Z Z) \times_{Y \times_W Z} (0 \times_0 0) \simeq (X \times_Y 0) \times_{Z \times_W 0} (Z \times_Z 0) \simeq \mathrm{fib}(f) \times_{\mathrm{fib}(f')} 0.$$

Similarly, the total cofibre $\mathrm{tocofib} \coloneqq \mathrm{cofib}(Z \cup_X Y \to W)$ is equivalently computed as $\mathrm{cofib}(\mathrm{cofib}(f) \to \mathrm{cofib}(f'))$. Thus, by the equivalences

$$\mathrm{cofib}(f) \simeq \Sigma\,\mathrm{fib}(f) \qquad \mathrm{cofib}(f') \simeq \Sigma\,\mathrm{fib}(f') \qquad \mathrm{fib}(Z \cup_X Y \to W) \simeq \Omega \mathrm{tocofib},$$

we obtain the required equivalence (6.22) as claimed.

Definition 6.6.3 ([LT19, Def. 2.5])**.** A spectrum valued localising invariant E is said to be k*–connective* if for every map $A \to B$ of connective $\mathbb{E}_1$–rings that is n–connective for some $n \geq 1$ (in particular, it is a π_0–isomorphism), the induced map $E(A) \to E(B)$ is $(n+k)$–connective.

Theorem 6.6.4 (Abstract quantitative descent, [LT19, Thm. 2.7])**.** *Suppose we have a pullback of $\mathbb{E}_1$–rings as in (6.5) where all rings are connective, and E is a k–connective spectrum–valued localising invariant. If $A' \otimes_A B \to B'$ is n–connective for some $n \geq 1$, then*

$$\begin{tikzcd} E(A) \arrow[r] \arrow[d] & E(B) \arrow[d] \\ E(A') \arrow[r] & E(B') \end{tikzcd}$$

is $(n+k-1)$*–cartesian.*

Proof. By Theorem 6.5.4, we get a pullback square

$$\begin{tikzcd} E(A) \arrow[r] \arrow[d] & E(B) \arrow[d] \\ E(A') \arrow[r] & E(A' \odot_A^{B'} B) \end{tikzcd}$$

of spectra. Thus, by Observation 6.6.2, we are left to argue that $E(A' \odot_A^{B'} B) \to E(B')$ is $(n+k)$–connective. But this is precisely supplied by the hypothesis on E. ☐

Now we want to fit K–theory into the situation above to obtain a quantitative descent statement for K–theory.

Theorem 6.6.5 ([BGT13, Thm. 9.53])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})$ *be connective. Then the ring map* $R \to \pi_0 R$ *induces an equivalence* $\tau_{\leq 0}\mathbb{K}(R) \xrightarrow{\simeq} \tau_{\leq 0}\mathbb{K}(\pi_0 R)$.

Construction 6.6.6 ([BGT13, pg. 819])**.** Let $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$ which is connective. Let $\mathrm{GL}_n(R) \coloneqq \mathrm{Aut}_R(R^{\oplus n})$ and $M_n \coloneqq \mathrm{Map}_R(R^{\oplus n}, R^{\oplus n})$ be the $\mathbb{E}_1$–anima of automorphisms and endomorphisms of $R^{\oplus n}$ in $\mathrm{LMod}(R)$, respectively, where $\mathrm{GL}_n(R)$ is even a grouplike $\mathbb{E}_1$–anima. Then observe that we have a pullback of $\mathbb{E}_1$–anima

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M_n R \arrow[d] \\ \mathrm{GL}_n(\pi_0 R) \arrow[r] & M_n(\pi_0 R) \simeq \pi_0 M_n R. \end{tikzcd}$$

Moreover, there are natural maps $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ and $M_n(R) \to M_{n+1}(R)$ which are compatible, induced by $- \oplus R$. We then define $\mathrm{GL}(R) \coloneqq \mathrm{colim}_n \mathrm{GL}_n(R)$ and $M(R) \coloneqq \mathrm{colim}_n M_n R$, and so by the compatibility aforementioned, we obtain a map of $\mathbb{E}_1$–monoids

$$\mathrm{GL}(R) \longrightarrow M(R).$$

Now note that, writing $\mathrm{Free}_R \subseteq \mathrm{LMod}(R)$ for the full subcategory spanned by the finite direct sums of R, the anima $\coprod_{n\geq 0} B\mathrm{GL}_n(R) \simeq (\mathrm{Free}_R)^{\simeq}$ naturally attains an $\mathbb{E}_\infty$–monoid structure using the direct sum. Thus, by Theorem 3.2.21, we obtain that

$$(\mathbb{Z} \times B\mathrm{GL}(R))^+ \simeq (\coprod_{n\geq 0} B\mathrm{GL}_n(R))[(-\oplus R)^{-1}]^+ \simeq (\coprod_{n\geq 0} B\mathrm{GL}_n(R))^{\mathrm{gp}}$$

is an $\mathbb{E}_\infty$–group. In particular, $B\mathrm{GL}(R)^+$ is also an $\mathbb{E}_\infty$–group, i.e. a connective spectrum.

Theorem 6.6.7 ([BGT13, Lem. 9.39])**.** *Let* $R \in \mathrm{Alg}(\mathrm{Sp})$ *be connective. Then there is an equivalence* $\Omega^\infty \mathrm{K}(R) \simeq \mathrm{K}_0(\pi_0 R) \times \mathrm{BGL}(R)^+$ *in* $\mathrm{CGrp} \simeq \mathrm{Sp}_{\geq 0}$*. In particular, we have an equivalence* $\Omega^\infty \tau_{\geq 1}\mathbb{K}(R) \simeq \Omega^\infty \tau_{\geq 1}\mathrm{K}(R) \simeq \mathrm{BGL}(R)^+$.

Theorem 6.6.8 ("K–theory bumps up connectivity by 1", [LT19, Lem. 2.4])**.** *Suppose $A \to B$ is a map of connective $\mathbb{E}_1$–rings which is n–connective for some $n \geq 1$ (in particular, it is a π_0–isomorphism). Then the map $\mathbb{K}(A) \to \mathbb{K}(B)$ is $(n+1)$–connective.*

Proof. Since $\tau_{\leq 0}\mathbb{K}(-)$ of R depends only on $\pi_0 R$ by Theorem 6.6.5, it thus suffices to show the statement for $\tau_{\geq 1}\mathbb{K}(-) \simeq \tau_{\geq 1}\mathrm{K}(-)$. By Theorem 6.6.7, we have equivalences

$$\Omega^\infty \tau_{\geq 1}\mathbb{K}(A) \simeq \mathrm{BGL}(A)^+ \qquad \Omega^\infty \tau_{\geq 1}\mathbb{K}(B) \simeq \mathrm{BGL}(B)^+.$$

By inspecting the construction of GL in terms of pullbacks and using that $\pi_0 A \to \pi_0 B$ is an isomorphism, we obtain an equivalence

$$\mathrm{fib}\big(\mathrm{GL}(A) \to \mathrm{GL}(B)\big) \xrightarrow{\simeq} \mathrm{fib}\big(M(A) \to M(B)\big).$$

Since $\mathrm{GL} \simeq \Omega\mathrm{BGL}$ because GL is already grouplike, and since $\pi_* M(R) \simeq M(\pi_* R)$, the equivalence above and our hypothesis implies that the map

$$\mathrm{BGL}(A) \longrightarrow \mathrm{BGL}(B)$$

is $(n+1)$–connective, i.e. the fibre F has $\pi_i F = 0$ for $i \leq n$. In particular, since $n \geq 1$, this means that F is simply connected and so we may apply the Hurewicz theorem to get that $\widetilde{H}_i(F;\mathbb{Z}) = 0$ for $i \leq n$. Thus, by a simple application of the Serre spectral sequence, we get that

$$H_i(\mathrm{BGL}(A);\mathbb{Z}) \longrightarrow H_i(\mathrm{BGL}(B);\mathbb{Z})$$

is an isomorphism for $i \leq n$ and a surjection for $i = n+1$. Since the plus–construction is a homology isomorphism by Proposition 3.2.19 (1), we thus get that

$$H_i(\mathrm{BGL}(A)^+;\mathbb{Z}) \longrightarrow H_i(\mathrm{BGL}(B)^+;\mathbb{Z})$$

is an isomorphism for $i \leq n$ and a surjection for $i = n+1$. Hence, if we write $G = \mathrm{fib}(\mathrm{BGL}(A)^+ \to \mathrm{BGL}(B)^+)$, then by standard Serre class arguments using the Serre spectral sequence for the defining fibre sequence for G, we get that $H_i(G) = 0$ for $i \leq n$. Now, since $\mathrm{BGL}(A)^+$ and $\mathrm{BGL}(B)^+$ are infinite loop spaces and the map $\mathrm{BGL}(A)^+ \to \mathrm{BGL}(B)^+$ is a map in CGrp, we thus get that G is also naturally an object in CGrp. In particular, this means that $\pi_1 G \cong H_1(G) = 0$, and so we may apply the Hurewicz theorem to obtain that G is $(n+1)$–connective. Hence, the map $\Omega^\infty \tau_{\geq 1}\mathbb{K}(A) \simeq \mathrm{BGL}(A)^+ \to \mathrm{BGL}(B)^+ \simeq \Omega^\infty \tau_{\geq 1}\mathbb{K}(B)$ is $(n+1)$–connective, as desired. □

Theorem 6.6.9 (Quantitative K–theory descent, [LT19, Thm. A = Thm. 2.8])**.** *Suppose we have a pullback of $\mathbb{E}_1$–rings as in (6.5) where all rings are connective. If the map $A' \otimes_A B \to B'$ is n–connective for some $n \geq 1$, then the induced diagram*

$$\begin{tikzcd}
\mathbb{K}(A) \arrow[r] \arrow[d] & \mathbb{K}(B) \arrow[d] \\
\mathbb{K}(A') \arrow[r] & \mathbb{K}(B')
\end{tikzcd}$$

is n–cartesian.

Proof. Immediate combination of Theorem 6.6.4 and that K–theory is 1–connective by Theorem 6.6.8. ☐

6.7 Examples and complements on the circle–dot ring

It is a fact (c.f. Remark after Main Theorem in [LT19]) that if $A \to B$ and $A \to A'$ are maps of $\mathbb{E}_k$–algebras, then the tensor product $A' \otimes_A B$ carries a natural $\mathbb{E}_{k-1}$–algebra structure. In particular, when $k \geq 2$, we see that $A' \otimes_A B$ carries a natural $\mathbb{E}_1$–algebra structure. In this section, we explore the divergence of this $\mathbb{E}_1$–algebra structure from the $\mathbb{E}_1$–algebra structure on the circle–dot ring $A' \odot_M B$ from Theorem 6.5.4.

We first work out an instructive sample calculation of the circle–dot ring which illustrates that, as rings, the circle–dot ring is *different* from the tensor of rings, even though they are equivalent as bimodules. The idea here is that, even though the circle–dot ring is mysterious in general, we may use the fibre sequences in Proposition 6.3.12 to figure out enough of its multiplicative structures by sandwiching it between objects with better understood such structures.

Example 6.7.1 ([LT19, §4]). Let k be a discrete commutative unital ring and let α be an element of k. The obvious ring maps then give us a pullback of $\mathbb{E}_\infty$–rings (which may be checked on underlying k–modules, for which it is clear by checking on the associated long exact sequences of homotopy groups)

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[y] \arrow[d] \\
k[x] \arrow[r] & k[x,y]/(yx-\alpha).
\end{tikzcd} \tag{6.23}$$

Thus, by Theorem 6.5.4 we obtain an $\mathbb{E}_1$–ring $k[x] \odot_{k[x,y]/(yx-\alpha)} k[y]$ with underlying spectrum $k[x] \otimes_k k[y]$. The following calculation shows that it is *not*, however, equivalent to $k[x] \otimes_k k[y]$ as $\mathbb{E}_1$–rings.

First of all, observe that, since the ring map $k \to k[x]$ is flat (the underlying k–module of $k[x]$ is just $\bigoplus_{i\geq 0} k$), the $\mathbb{E}_1$–ring $k[x] \odot_{k[x,y]/(yx-\alpha)} k[y]$ is discrete with underlying k–module $k[x] \otimes_k k[y]$. Secondly, note that in the $\mathbb{E}_\infty$–ring $k[x] \otimes_k k[y]$, we have the datum of an equivalence $(x \otimes 1) \cdot (1 \otimes y) \simeq (1 \otimes y) \cdot (x \otimes 1)$. In particular, we get that $(x \otimes 1) \cdot (1 \otimes y) - (1 \otimes y) \cdot (x \otimes 1)$ has the property of being 0 (even though we have forgotten the *datum* of this equivalence!) in the underlying $\mathbb{E}_1$–ring of $k[x] \otimes_k k[y]$. We show now that $(x \otimes 1) \cdot (1 \otimes y) - (1 \otimes y) \cdot (x \otimes 1)$ fails to be equivalent to 0 in the $\mathbb{E}_1$–ring $k[x] \odot_{k[x,y]/(yx-\alpha)} k[y]$.

To see this, since by Theorem 6.5.4 we know that $k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]$ is equivalent to $k[x]\otimes_k k[y]$ as $(k[x],k[y])$–bimodules, we know that $(x\otimes 1)\cdot(1\otimes y)\simeq x\otimes y$ in $k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]$. To compute $(1\otimes y)\cdot(x\otimes 1)$, we need a sufficient understanding of $k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]$ as a $k[y]$–left module. So let I be the fibre of the right vertical map in (6.23). By Proposition 6.3.12, we have the (rotated) cofibre sequence of $(k[y],k[y])$–bimodules

$$k[y]\longrightarrow k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]\xrightarrow{j}\Sigma I\otimes_k k[y].$$

Representing this as chain complexes, this cofibre sequence may be rewritten as the following diagram of complexes of left $k[y]$–modules

$$\begin{tikzcd}
0 \arrow[d] & {} \arrow[r, "\mathrm{can}\otimes\mathrm{id}"] & {} & 0 \arrow[d] & {} \arrow[r, "j"] & {} & k[y]\otimes_k k[y] \arrow[d] \\
k[y] & & & k[x]\otimes_k k[y] & & & k[x,y]/(yx-\alpha)\otimes_k k[y].
\end{tikzcd}$$

Since j was a map of $k[y]$–left modules, we learn that

$$j((1\otimes y)\cdot(x\otimes 1))=(y\otimes 1)\cdot(x\otimes 1)=yx\otimes 1=\alpha\otimes 1=j\big(\alpha(1\otimes 1)\big).$$

Thus, we see that in $k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]$,

$$(1\otimes y)\cdot(x\otimes 1)=\alpha(1\otimes 1)+1\otimes f(y)$$

for some $f(y)\in k[y]$. In particular, $(1\otimes y)\cdot(x\otimes 1)-(x\otimes 1)\cdot(1\otimes y)=(1\otimes y)\cdot(x\otimes 1)-x\otimes y\neq 0$ in $k[x]\odot_{k[x,y]/(yx-\alpha)}k[y]$, whose underlying k–module is $k[x]\otimes_k k[y]$, as claimed.

Remark 6.7.2. Land–Tamme proved something stronger in [LT19, Prop. 4.1], namely they identified the whole $\mathbb{E}_1$–ring structure of the circle–dot ring as $k\langle x,y\rangle/(yx-\alpha)$ where $k\langle x,y\rangle$ is the free noncommutative polynomial ring on two generators. But I did not manage to understand their arguments (specifically, they claimed that the map j was injective), so I have refrained from stating the full result in the example above.

Next, we record the following complementary situation where the circle–dot ring *is* given by the tensor of rings. This was exploited by [LT23] in their Examples 2.19 and 2.20 to obstruct the existence of higher commutative refinements of the square of $\mathbb{E}_1$–rings (6.21).

Proposition 6.7.3 ([LT23, Prop. 2.18])**.** *Let (A',B,M) be a Milnor context. Suppose that the diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & A'\odot_M B
\end{tikzcd}$$

from (6.21) refines to a diagram of $\mathbb{E}_2$*–*k*–algebras. Then* $A' \otimes_A B$ *naturally attains an* $\mathbb{E}_1$*–*k*–algebra and this is equivalent to* $A' \odot_M B$ *as* $\mathbb{E}_1$*–*k*–algebras.*

Proof. Suppose that the diagram commutes as $\mathbb{E}_2$–k–algebras. Then the $\mathbb{E}_2$–map $A \to A'$ makes A' an $\mathbb{E}_1$–A–algebra: this is since it induces an adjunction

$$- \otimes_A A' \colon \mathrm{Mod}(A) \rightleftharpoons \mathrm{Mod}(A') : \mathrm{fgt}$$

where the left adjoint refines to an $\mathbb{E}_1$–monoidal functor and hence the right adjoint becomes lax $\mathbb{E}_1$–monoidal. Applying $\mathrm{Alg}_{\mathbb{E}_1}(-)$ to this adjunction and extracting the adjunction unit then gives A' as an object in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(A))$. Similar arguments show that we may view the $\mathbb{E}_2$–map $A' \to A' \odot_M B$ as a morphism in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(A))$. By similar arguments again, we obtain an adjunction $- \otimes_A B \colon \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(A)) \rightleftharpoons \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(B)) : \mathrm{fgt}$ which gives $A' \otimes_A B$ a canonical $\mathbb{E}_1$–B–algebra structure. Moreover, since we may also view $A' \odot_M B$ as an object in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(B))$, the adjunction above extends the morphism $A' \to A' \odot_M B$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(A))$ to a morphism $A' \otimes_A B \to A' \odot_M B$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}(B))$. But by Proposition 6.3.15, the morphism $A' \otimes_A B \to A' \odot_M B$ is an equivalence on underlying k–modules. Hence, it must also be an equivalence as $\mathbb{E}_1$–k–algebras, as was to be shown. □

6.8 Application: the theory of truncating invariants

The following is the key example of interest in this section:

Example 6.8.1 ([LT19, Proof of Thm. 2.25]). Let A be a discrete unital ring and $I \subseteq A$ a square–zero two–sided ideal, i.e. $I^2 = 0$. Consider the pullback of $\mathbb{E}_1$–rings

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A/I \arrow[d, "\mathrm{incl}\oplus\mathrm{can}"] \\
A/I \arrow[r, "\mathrm{incl}\oplus 0"'] & A/I \oplus \Sigma I
\end{tikzcd}$$

coming from the theory of square–zero extensions (c.f. [Lur17, §7.4.1]).

We claim that the map $A/I \odot_{A/I\oplus\Sigma I} A/I \to A/I \oplus \Sigma I$ exhibits the target as a 1–truncation of the source. To see this, note first that both sides are connective since the underlying spectrum of the source is given by $A/I \otimes_A A/I$ by Proposition 6.3.12 and the target is clear. In fact, it is also clear that the target is 1–truncated. Now, by Observation 6.3.14, we obtain a map of cofibre sequences

$$\begin{tikzcd}
I \otimes_A A/I \arrow[r] \arrow[d] & A/I \arrow[r] \arrow[d, equal] & A/I \odot_{A/I\oplus\Sigma I} A/I \arrow[d] \\
I \arrow[r] & A/I \arrow[r] & A/I \oplus \Sigma I.
\end{tikzcd}$$

Since $\pi_1 I = 0$, the left vertical map is clearly π_1–surjective. Furthermore, since $I^2 = 0$, the right exact sequence of abelian groups $I \otimes_A^{\heartsuit} I \xrightarrow{0} I \otimes_A^{\heartsuit} A = I \to I \otimes_A^{\heartsuit} A/I \to 0$ shows that the map $I \to I \otimes_A^{\heartsuit} A/I$ is an isomorphism, and thus $\pi_0(I \otimes_A A/I) \cong I \otimes_A^{\heartsuit} A/I \to \pi_0 I \cong I$ is an isomorphism. All in all, by the 5–lemma, we see that the right vertical map is a π_0 and π_1 isomorphism, as claimed.

Definition 6.8.2 ([LT19, Def. 3.1])**.** Let E be a localising invariant. We call it:

- *truncating* if for any connective ring spectrum R, the map $E(R) \to E(\pi_0 R)$ is an equivalence,
- *nilinvariant* if for every nilpotent two–sided ideal $I \subseteq A$ in a discrete unital ring A, the canonical map $E(A) \to E(A/I)$ is an equivalence,
- *excisive* if it sends squares of $\mathbb{E}_1$–rings

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd} \tag{6.24}$$

  to pullback squares, provided the square of rings satisfies the following two conditions:

  (E1) the square (6.24) is cartesian and all $\mathbb{E}_1$–rings in it are connective,

  (E2) the induced map $\pi_0(A' \otimes_A B) \to \pi_0 B'$ is an isomorphism.

Theorem 6.8.3 (Truncating invariants are excisive and nilinvariant, [LT19, Thm. 3.3, Cor. 3.5])**.** *Any truncating is excisive and nilvariant.*

Proof. Let E be a truncating invariant. To see excision, suppose we have a square (6.24) of $\mathbb{E}_1$–rings satisfying (E1) and (E2), then $E(A' \odot_{B'} B) \simeq E(\pi_0 A' \odot_{B'} B) \to E(\pi_0 B') \simeq E(B')$ is an equivalence. Hence, by Theorem 6.5.4, we see that E sends (6.24) to a pullback, as required.

To see nilinvariance, let $n > 1$ be the smallest number such that $I^n = 0$. Note that for each k, $J_k \coloneqq \ker(A = A/I^k \to A/I^{k-1})$ is a square–zero two–sided ideal of A/I^k. Hence, if we can deal with the case $n = 2$, then all the maps in

$$E(A) = E(A/I^n) \longrightarrow E(A/I^{n-1}) \longrightarrow \cdots \longrightarrow E(A/I^2) \longrightarrow E(A/I)$$

are equivalences, thus implying the case for general $n > 1$. To deal with the case $n = 2$, we apply that E is excisive from the first part to the pullback square in Example 6.8.1 together with the fact that $A/I \odot_A^{A/I \oplus \Sigma I} A/I \to A/I \oplus \Sigma I$ is a π_0–isomorphism by Example 6.8.1 and that E is truncating to obtain a pullback square

$$\begin{tikzcd} E(A) \arrow[r] \arrow[d] & E(A/I) \arrow[d] \\ E(A/I) \arrow[r] & E(A/I \oplus \Sigma I). \end{tikzcd}$$

Again, using that E is truncating and that $A/I \to A/I \oplus \Sigma I$ is a π_0–isomorphism yields that the bottom map is an equivalence, whence the top horizontal map is an equivalence too, as desired. □

Proposition 6.8.4 (A circle–dot Blakers–Massey principle, [LMM+20, Lem. 3.2])**.** *Suppose we have a pullback of rings as in (6.5) where A is moreover connective. If $A \to A'$ is n–connective and $A \to B$ is m–connective, then the induced map $A' \odot_A^{B'} B \to B'$ is $(m+n+2)$–connective.*

Proof. Recall the notation for the fibres from Notation 6.3.11. By the identification of $A' \odot_A^{B'} B$ in Proposition 6.3.6, we have a map of cofibre sequences

$$\begin{tikzcd}
I \otimes_A B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & A' \odot_A^{B'} B \arrow[d] \\
I \arrow[r] & B \arrow[r] & B'
\end{tikzcd}$$

and hence an identification

$$\mathrm{fib}\left(A' \odot_A^{B'} B \to B'\right) \simeq \Sigma\,\mathrm{fib}(I \otimes_A B \xrightarrow{\mu} I) \tag{6.25}$$

where μ is given by the B–right module structure on I. Note that μ has a section $\sigma\colon I \to I \otimes_A B$ induced by the map $A \to B$. By definition, we have $\mathrm{fib}(\sigma) \simeq I \otimes_A J$ which is $(n+m)$–connective as A is connective. In other words, σ is an isomorphism in degrees $\leq m+n-1$ and surjective in degree $m+n$. On the other hand, $\mu\circ\sigma \simeq \mathrm{id}_I$, and so μ is surjective in every degree and σ is injective in every degree. In particular, together with our connectivity conclusions about σ above, we see that μ is an isomorphism in degrees $\leq m+n$ and surjective in degree $m+n+1$, i.e. μ is $(m+n-1)$–connective. Thus, by the identification from (6.25) we see that $A' \odot_A^{B'} B \to B'$ is $(m+n+2)$–connective as desired. □

Recall the notion of M–acyclity with respect to $M \in \mathrm{Sp}$ from Construction 2.5.18.

Definition 6.8.5. Let M be a spectrum and E a localising invariant. We say that E is *truncating on M–acyclic ring spectra* if it is truncating on M–acyclic connective ring spectra.

Observation 6.8.6. Note that if R is M–acyclic, then so is $\tau_{\leq k}R$ for all k since we have a ring map $L_M R \to L_M \tau_{\leq k} R$.

Lemma 6.8.7 (Rigidity of acyclic truncating invariants, [LMM+20, Lem. 3.3])**.** *Let E be a localising invariant. Suppose there exists a $k \geq 0$ with $E(R) \to E(\tau_{\leq k}R)$ being an equivalence for any M–acyclic connective ring spectrum R. Then E is truncating on M–acyclic ring spectra.*

Proof. If will suffice by induction to show that for $k > 0$ as in the hypothesis, the map $E(\tau_{\leq k}R) \to E(\tau_{\leq k-1}R)$ is an equivalence for all M–acyclic connective ring spectra. For this, first recall that $\tau_{\leq k}R \to \tau_{\leq k-1}R$ is a square–zero extension. Hence, as in Example 6.8.1, by [Lur17, Prop. 7.4.1.29] there is a pullback of rings

$$\begin{tikzcd}
\tau_{\leq k}R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_{\leq k-1}R \arrow[d, "\mathrm{incl}\oplus\mathrm{can}"] \\
\tau_{\leq k-1}R \arrow[r, "\mathrm{incl}\oplus 0"] & \tau_{\leq k-1}R \oplus \Sigma^{k+1}\pi_k R
\end{tikzcd}$$

Note that all ring spectra in the square are connective and M–acyclic. By Theorem 6.5.10, we have a pullback of spectra

$$\begin{tikzcd}
E(\tau_{\leq k}R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(\tau_{\leq k-1}R) \arrow[d] \\
E(\tau_{\leq k-1}R) \arrow[r] & E\Big(\tau_{\leq k-1}R \odot_{\tau_{\leq k-1}R\oplus\Sigma^{k+1}\pi_k R} \tau_{\leq k-1}R\Big).
\end{tikzcd} \tag{6.26}$$

Hence, we want to show that the right vertical map is an equivalence. Consider the commuting triangle of rings

$$\begin{tikzcd}
\tau_{\leq k-1}R \arrow[d] \arrow[dr, "\mathrm{incl}\oplus\mathrm{can}"] & \\
\tau_{\leq k-1}R \odot \tau_{\leq k-1}R \arrow[r] & \tau_{\leq k-1}R \oplus \Sigma^{k+1}\pi_k R.
\end{tikzcd}$$

By the Blakers–Massey principle Proposition 6.8.4, the bottom map is $(2k+2)$–connective and so in particular, an equivalence upon applying $\tau_{\leq k}$. On the other hand, the diagonal map is also an equivalence upon applying $\tau_{\leq k}$. Hence, the vertical map is a $\tau_{\leq k}$–equivalence. Furthermore, since the circle–dot ring has underlying spectrum $\tau_{\leq k-1}R \otimes_{\tau_{\leq k}R} \tau_{\leq k-1}R$, it is also M–acyclic. Thus, by our hypothesis on E, the right vertical map in (6.26) is an equivalence, and so the left vertical is an equivalence. Consequently, E satisfies that $E(R) \to E(\tau_{\leq k-1}R)$ is an equivalence for all M–acyclic ring spectrum R. Thus, by induction, we obtain that $E(R) \to E(\pi_0 R)$ is an equivalence for all M–acyclic ring spectrum R as required. □

Remark 6.8.8. The proof we have presented here is slightly modified from that of Land–Mathew–Meier–Tamme since they used a slightly different pullback square from square zero extension theory which is not the one immediately supplied by [Lur17, Prop. 7.4.1.29].

7 A recent celebrated result: chromatic purity

7.1 Prelude: chromatic homotopy theory

In the following we will fix a prime number p and work in the ∞-category of p-local spectra $\mathrm{Sp}_{(p)}$.

Small world and K

Recollections 7.1.1 (Morava K-theories)**.** For every $n \geq 1$, let $K(n)$ denote the nth Morava K-theory spectrum with

$$\pi_* K(n) \simeq \mathbb{F}_p[v_n^{\pm 1}],$$

where $|v_n| = 2(p^n - 1)$. By convention, set $K(0) = \mathrm{H}\mathbb{Q}$ and $K(\infty) = \mathrm{H}\mathbb{F}_p$. For $n \geq 1$, the spectrum $K(n)$ admits the structure of an $\mathbb{E}_1$-ring spectrum and there is a map of spectra $\mathrm{MU}_{(p)} \to K(n)$ which equips $K(n)$ with a complex orientation. The associated formal group law has height exact n.

Definition 7.1.2. Let X be a finite p-local spectrum and let $n \geq 1$.

1. The spectrum X has type $\geq n$ if $K(i) \otimes X \simeq 0$ for every $i < n$.
2. The spectrum X has type n if it has type $\geq n$ and $K(n) \otimes X$ is nonzero.

Let $\mathrm{Sp}^{\omega}_{\mathrm{ht} \geq n}$ denote the full subcategory of the ∞-category $\mathrm{Sp}^{\omega}_{(p)}$ of finite p-local spectra spanned by those which are of type $\geq n$.

Exercise 7.1.3. Let $\mathbb{S}_{(p)}$ denote the p-local sphere spectrum.

1. Prove that $\mathbb{S}_{(p)}$ is of type ≥ 0 but not of type ≥ 1.
2. Let $\mathbb{S}_{(p)}/p$ denote the cofiber (formed in Sp) of the map given by multiplication by p. Show that $\mathbb{S}_{(p)}/p$ is of type ≥ 1 but not of type ≥ 2.

Exercise 7.1.4. Let $X \in \mathrm{Sp}^{\omega}_{(p)}$. Prove that if X is nonzero, then X there is an n such that $K(n) \otimes X \neq 0$. ***Hint:*** *Observe that for $i \gg 0$, the Atiyah–Hirzebruch spectral sequence for $K(i)$ degenerates for degree reasons for a fixed finite spectrum X.*

Exercise 7.1.5. Prove that $\mathrm{Sp}^{\omega}_{\mathrm{ht} \geq n}$ is a thick subcategory of $\mathrm{Sp}^{\omega}_{(p)}$.

Remark 7.1.6. Let X be a finite p-local spectrum. If $K(n)\otimes X \simeq 0$, then $K(i)\otimes X \simeq 0$ for $i < n$ (cf. [Rav84, Theorem 2.11]). Consequently, we may equivalently have defined a finite p-local spectrum X to be of type $\geq n$ if $K(n-1)\otimes X \simeq 0$. Note that it also follows that if X is of type n, then $K(i)\otimes X$ is nonzero for $i > n$. We remark that the cited result [Rav84] crucially relies on the assumption that X is finite, in which case the result ensures that the so-called support set

$$\mathrm{supp}(X) = \{i \geq 0 \mid K(i)\otimes X \neq 0\}$$

is the ray $\{i_0, i_0+1, \ldots\}$, where i_0 is the smallest natural number such that $K(i)\otimes X$ is nonzero. For a general p-local spectrum the support set of X might not be a ray.

It follows from Remark 7.1.6 that we obtain a sequence of thick subcategories

$$\cdots \hookrightarrow \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n+1} \hookrightarrow \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n} \hookrightarrow \cdots \hookrightarrow \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq 1} \hookrightarrow \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq 0} = \mathrm{Sp}^{\omega}_{(p)}.$$

Mitchel proved that these inclusions are proper (cf. [Mit85]), namely that for every $n \geq 0$, there exists a finite p-local spectrum of type n. We will later deduce this from the periodicity theorem of Hopkins–Smith [HS98]. Furthermore, Hopkins–Smith [HS98] proved that these thick subcategories exhaust the thick subcategories of the ∞-category of p-local finite spectra:

Theorem 7.1.7 (Thick subcategory theorem)**.** *If $\mathcal{C}$ is a thick subcategory of $\mathrm{Sp}^{\omega}_{(p)}$ which is nonzero, then $\mathcal{C} = \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n}$ for some $n \geq 0$.*

To end this primer on the $K(n)$ spectra, we record the following extremely powerful theorem of Hopkins–Smith which says that the $K(n)$'s jointly detect nilpotence. This is a vast generalisation of Nishida's famous nilpotence result that says that all elements of $\pi_*\mathbb{S}$ where $* > 0$ are nilpotent.

Theorem 7.1.8 (Nilpotence theorem, [HS98, Thm. 3 (i)])**.** *Let R be a p–local ring spectrum. Then an element $\alpha \in \pi_* R$ is nilpotent if and only if for all $0 \leq n \leq \infty$, the image $\alpha \in \pi_*(K(n)\otimes R)$ is nilpotent.*

Big world

We will next study the filtration discussed above on the entire ∞-category of p-local spectra without imposing any finiteness conditions.

Notation 7.1.9. Let $\mathrm{Sp}_{\mathrm{ht}\geq n}$ denote the smallest stable subcategory of $\mathrm{Sp}_{(p)}$ which contains $\mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n}$ and which is closed under colimits. In other words, $\mathrm{Sp}_{\mathrm{ht}\geq n}$ is the smallest localising subcategory of $\mathrm{Sp}_{(p)}$ containing $\mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n}$.

As before, we obtain a filtration of the ∞-category of p-local spectra

$$\cdots \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq n+1} \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq n} \hookrightarrow \cdots \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq 1} \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq 0} = \mathrm{Sp}_{(p)}$$

and the goal of chromatic homotopy theory is to understand the graded pieces.

Definition 7.1.10. The category Sp_{v_n} of v_n-periodic spectra is the Verdier quotient

$$\mathrm{Sp}_{v_n} = \mathrm{Sp}_{\mathrm{ht}\geq n}/\mathrm{Sp}_{\mathrm{ht}\geq n+1}$$

for each $n \geq 1$.

This means that Sp_{v_n} is a stable category which receives an exact functor $\mathrm{Sp}_{\mathrm{ht}\geq n} \to \mathrm{Sp}_{v_n}$ such that for every stable ∞-category $\mathcal{E}$, the restriction functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}_{v_n}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}_{\mathrm{ht}\geq n}, \mathcal{E})$$

is fully faithful whose essential image is spanned by those exact functors which vanish after precomposition with $\mathrm{Sp}_{\mathrm{ht}\geq n+1} \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq n}$. This is a somewhat abstract characterisation and we will now explain how one can obtain a more concrete description of Sp_{v_n} using the periodicity theorem of Hopkins–Smith [HS98].

Definition 7.1.11. Let X be a finite p-local spectrum and let $n \geq 1$. A v_n-self map is a map $v\colon \Sigma^d X \to X$ for some integer $d \geq 0$, satisfying the following properties:

1. The induced map on $K(n)$-homology $K(n)_*(X) \to K(n)_*(X)$ is an isomorphism.
2. For $i \neq n$, the induced map $K(i)_*(X) \to K(i)_*(X)$ is nilpotent.

A v_0-self map is a map $X \to X$ which acts by multiplication with a rational number on $K(0)_*(X) = H_*(X, \mathbb{Q})$.

Definition 7.1.12. A pointed anima X is of type $\geq n$ if $\Sigma^\infty X$ is of type $\geq n$ in the sense of Definition 7.1.2 above. A v_n-self map of X is a map v such that $\Sigma^\infty v$ is a v_n-self map of $\Sigma^\infty X$.

Multiplication by p is an example of a v_0-self map of $\mathbb{S}$. The cofiber $\mathbb{S}/p$ is of type 1 and, bulding on work of Barratt, Adams [Ada66] constructed a v_1-self map on $\mathbb{S}/p$. The famous periodicity theorem of Hopkins–Smith [HS98] ensures that we can inductively continue this procedure:

Theorem 7.1.13 (The periodicity theorem)**.** *Let $n \geq 1$. The following holds:*

1. *If X is a finite p-local spectrum of type $\geq n$, then X admits a v_n-self map.*
2. *If $f\colon X \to Y$ is a map of finite p-local spectra with v_n-self maps $v\colon \Sigma^d X \to X$ and $w\colon \Sigma^e Y \to Y$, respectively, then there exists $M, N \gg 0$ such that the square*

$$\begin{tikzcd}
\Sigma^{Md} X \arrow[r, "f"] \arrow[d, "v^M"] & \Sigma^{Ne} Y \arrow[d, "w^N"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes.

Exercise 7.1.14. Let X be a finite p-local spectrum. Prove that if v and w are v_n-self maps of X, then there exists $M, N \gg 0$ such that $v^M \simeq w^N$.

Remark 7.1.15. It follows from Theorem 7.1.13 that the thick subcategories $\mathrm{Sp}_{\mathrm{ht}\geq n}$ are distinct. We explain how we can inductively construct a finite p-local spectrum of type n for every $n \geq 0$. First note that $\mathbb{S}_{(p)}$ is of type 0 and multiplication by p is a v_0-self map of $\mathbb{S}_{(p)}$. The cofiber $\mathbb{S}_{(p)}/p$ is of type 1 but not of type 2. By Theorem 7.1.13, the cofiber $\mathbb{S}_{(p)}/p$ supports a v_1-self map v_1 which we may use to form the spectrum $\mathbb{S}_{(p)}/(p, v_1)$ which is of type 2 but not type 3. Inductively, if X is of type n, then it supports a v_n-self map v_n such that X/v_n is of type $n+1$ but not type $n+2$.

Let V_n denote a pointed anima of type $n \geq 1$ and let v_n denote a v_n-self map of V_n.

Definition 7.1.16. For X a pointed anima or spectrum, the v_n-periodic homotopy groups of X with coefficients in V_n are defined by

$$v_n^{-1}\pi_*(X; V_n) = \mathbb{Z}[v_n^{\pm 1}] \otimes_{\mathbb{Z}[v_n]} \pi_*\mathrm{Map}_*(V_n, X).$$

for $n \geq 1$. The v_0-periodic homotopy groups of X are defined by

$$v_0^{-1}\pi_*(X) = \pi_*(X)[1/p].$$

Exercise 7.1.17. Consider the telescope

$$v_n^{-1}V_n^\vee = \mathrm{colim}(V_n^\vee \xrightarrow{v_n^\vee} \Sigma^{-d}V_n^\vee \xrightarrow{v_n^\vee} \cdots),$$

where $V_n^\vee$ denotes the Spanier–Whitehead dual of V_n. Prove that

$$v_n^{-1}\pi_*(X; V_n) \simeq \pi_*(v_n^{-1}V_n^\vee \otimes X).$$

The thick subcategory theorem ensures that the construction of the v_n-periodic homotopy groups is independent of the choice of V_n, and the asymptotic uniqueness of v_n-self maps ensures that it is independent of the choice of v_n.

Definition 7.1.18. A map of spectra $X \to Y$ is a v_n-periodic equivalence provided that $v_n^{-1}\pi_*(X) \to v_n^{-1}\pi_*(Y)$ is an isomorphism.

Terminology 7.1.19. Set $T(0) = \mathbb{S}[1/p]$. For every $n \geq 1$, the nth telescope is given by $T(n) = v_n^{-1}V_n^\vee$, where v_n denotes a v_n-self map of a finite anima V_n of type n. Let $\mathrm{Sp}_{T(n)}$ denote the full subcategory of spectra spanned by those which are $T(n)$-local via the Construction 2.5.18. This means that there is a symmetric monoidal Bousfield localisation

$$\begin{tikzcd}\mathrm{Sp} \arrow[r, shift left, "L_{T(n)}"] & \mathrm{Sp}_{T(n)} \arrow[l, shift left, hook]\end{tikzcd}$$

such that the following are satisfied:

1. The functor $L_{T(n)}$ inverts the $T(n)$-local equivalences, that is those morphisms f such that $T(n) \otimes f$ is an equivalence.

2. The functor $L_{T(n)}$ carries the $T(n)$-acyclic objects to zero, that is if $X \otimes T(n) \simeq 0$, then $L_{T(n)}X \simeq 0$.

3. Every $Y \in \mathrm{Sp}_{T(n)}$ is $T(n)$-local in the sense that $\mathrm{Map}_{\mathrm{Sp}}(X, Y) \simeq 0$ for every $T(n)$-acyclic spectrum X.

The unit of the Bousfield localisation provides a map $X \to L_{T(n)}X$ for every spectrum X which is a $T(n)$-local equivalence.

Terminology 7.1.20. Recall that L_n^f denotes localisation at $\mathbb{Q} \oplus T(1) \oplus \cdots \oplus T(n)$. Following [LMM+20, Def. 2.5], we denote by $L_n^{p,f}$ the localisation at $T(0) \oplus \cdots \oplus T(n)$. Note here that the difference between L_n^f and $L_n^{p,f}$ is that we have used $\mathbb{Q} \simeq \mathbb{S}_{(p)}[\frac{1}{p}]$ for the former whereas in the latter we used $T(0) = \mathbb{S}[\frac{1}{p}]$. Following [CMN+20, Rmk. 2.10], by periodic localisations we mean any one of the localisations $L_{T(n)}, L_{K(n)}, L_n^f, L_n^{p,f}, L_n$. It is a fact that $L_n^{p,f}$ and L_n^f are smashing localisations.

We can now describe the category of v_n-periodic spectra in terms of the v_n-self maps as promised:

Proposition 7.1.21. *The category Sp_{v_n} of v_n-periodic spectra is equivalent to:*

1. *The Dwyer–Kan localisation of $\mathrm{Sp}_{(p)}$ at the v_n-periodic equivalences.*

2. *The ∞-category $\mathrm{Sp}_{T(n)}$ of $T(n)$-local spectra.*

Proof. First note that (1) and (2) are equivalent since v_n-periodic equivalences equal the $T(n)_*$-equivalences by definition. We prove that Sp_{v_n} is equivalent to (1).

Note that if W is a finite spectrum of type $\geq n+1$, then $W \to 0$ is a v_n-periodic equivalence. It follows that the composite of exact functors

$$\mathrm{Sp}_{\mathrm{ht}\geq n+1} \hookrightarrow \mathrm{Sp}_{\mathrm{ht}\geq n} \hookrightarrow \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}[v_n^{-1}]$$

is equivalent to the zero functor. Thus, we obtain an exact functor $\mathrm{Sp}_{v_n} \to \mathrm{Sp}_{(p)}[v_n^{-1}]$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Sp}_{\mathrm{ht}\geq n} \arrow[r] \arrow[d] & \mathrm{Sp}_{(p)}[v_n^{-1}] \\
\mathrm{Sp}_{v_n} \arrow[ur, dashed] &
\end{tikzcd}$$

by the universal property of Sp_{v_n}. This functor admits an inverse since the composite

$$\mathrm{Sp}_{(p)} \xrightarrow{L_n^f} \mathrm{Sp}_{\mathrm{ht}\geq n} \to \mathrm{Sp}_{v_n}$$

canonically factors over the Dwyer–Kan localisation $\mathrm{Sp}_{(p)}[v_n^{-1}]$. □

The following exercise is a wonderful way to familiarize yourself with the definitions (cf. [LMM+20, Lemma 2.2]).

Exercise 7.1.22. Let X denote a spectrum.

(1) Prove that if X is bounded above, then $L_{T(n)}X \simeq 0$ for every $n \geq 1$.

(2) Prove that $\tau_{\geq k}X \to X$ is a $T(n)$-local equivalence for all k and every $n \geq 1$.

(3) Prove that $K(m)$ is $T(n)$-acyclic if $m \neq n$.

(4) Prove that if $X \otimes T(n) \simeq 0$, then $X \otimes K(n) \simeq 0$. ***Hint:*** *use that every module over $K(i)$ is a direct sum of shifted copies of $K(i)$.*

(5) Prove that if $X \otimes \mathbb{S}/p \simeq 0$, then $X \otimes T(n) \simeq 0$ for every $n \geq 1$.

(6) Prove that the p-completion map $X \to X_p^\wedge$ is a $T(n)$-local equivalence for every $n \geq 1$. ***Hint:*** *Show that the fiber of the p-completion map is $\mathbb{S}/p$-acyclic.*

Exercise 7.1.22 establishes two important features of chromatic homotopy theory. First of all, note that (2) ensures that we may always replace a spectrum with its highly connective cover for the purposes of studying a spectrum in $\mathrm{Sp}_{T(n)}$. Furthermore, for the purpose of chromatic homotopy theory, we may always replace a spectrum with its p-completion by virtue of (6).

Observation 7.1.23. Let $n \geq 1$. It will be useful to observe that we may always assume that $T(n)$ admits the structure of an $\mathbb{E}_1$-ring. Recall that $T(n)$ is defined by $T(n) = V_n[v^{-1}]$, where V_n is a finite spectrum of type n and v is a v_n-self map. We may replace V_n with $W_n = V_n \otimes DV_n = \mathrm{End}(V_n)$ which canonically carries the structure of an $\mathbb{E}_1$-ring and oncemore is of type n. The v_n-self map v defines a v_n-self map w of W_n. By [HS98, Thm. 11], a power of w lies in the center of $\pi_*(W_n)$, so the localisation $W_n[w^{-1}]$ admits the structure of an $\mathbb{E}_1$-ring.

The following result provides a converse to Exercise 7.1.22 (4) in the case of rings.

Lemma 7.1.24 ([LMM+20, Lem. 2.3])**.** *Let R be a ring spectrum and $n \geq 1$. Then R is $K(n)$–acyclic if and only if it is $T(n)$–acyclic.*

Proof. We have already seen the direction that $T(n)$-acyclic implies $K(n)$-acyclic in Exercise 7.1.22 above. As explained in Observation 7.1.23, we may assume that $T(n)$ carries the structure of an $\mathbb{E}_1$-ring. Now a ring spectrum like $T(n) \otimes R$ is zero if and only if its unit is nilpotent. By the Hopkins–Smith nilpotence Theorem 7.1.8, this is the case if and only if $K(m) \otimes T(n) \otimes R \simeq 0$ for all $0 \leq m \leq \infty$. If $m \neq n$, then $K(m)$ is $T(n)$–acyclic by Exercise 7.1.22. When $m = n$, we have that $K(n) \otimes T(n) \otimes R \simeq 0$ by hypothesis that R is $K(n)$–acyclic. Then $T(n) \otimes R \simeq 0$. □

Lemma 7.1.25. *The ∞-category of $L_n^{p,f}$-acyclic spectra is equivalent to* $\mathrm{Ind}(\mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n+1})$. *In particular, $L_n^{p,f}$ is smashing.*

Proof. We will give an indication of the proof (cf. [LMM+20, Lemma 2.6]). If $X \in \mathrm{Ind}(\mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n+1})$, then X can be written as a filtered colimit of finite p-local spectra of type $\geq n+1$ from which we conclude that X is $T(i)$-acyclic for every $i \leq n$ and thus $L_n^{p,f}$-acyclic. For the converse, we proceed by induction on n. First let $n = 0$ and assume that X is $T(0) = \mathbb{S}[p^{-1}]$-acyclic. There is a cofiber sequence

$$X \to X \otimes \mathbb{S}[p^{-1}] \to X \otimes \mathbb{S}/p^{\infty},$$

where $\mathbb{S}/p^{\infty} \simeq \mathrm{colim}_n \mathbb{S}/p^n$ which implies that $X \simeq \Sigma^{-1}(X \otimes \mathbb{S}/p^{\infty})$ by our assumption that X is $\mathbb{S}[p^{-1}]$-acyclic. We may write X as a filtered colimit of finite spectra and after tensoring with $\mathbb{S}/p^n$ this becomes a filtered colimit of finite spectra of type 1 as desired. The induction step is similar using the cofiber sequence

$$X \otimes \mathbb{S}/p^n \to X \otimes \mathbb{S}/p^n[v_1^{-1}] \to X \otimes \mathbb{S}/(p^n, v_1^{\infty})$$

to obtain the case of $n = 1$ and so on, where we employ the periodicity Theorem 7.1.13 to obtain v_n–self maps for $n \geq 1$. □

As a consequence, we obtain the following fracture square principle from a standard exercise, c.f. [LMM+20, Lem. 2.8] for the generic statement.

Lemma 7.1.26 (Periodic fracture square). *For every $0 \leq m < n$, there is a pullback of functors*

$$\begin{tikzcd}
L_n^{p,f} \arrow[r] \arrow[d] & L_{T(m+1)\oplus\cdots\oplus T(n)} \arrow[d] \\
L_m^{p,f} \arrow[r] & L_m^{p,f} L_{T(m+1)\oplus\cdots\oplus T(n)}
\end{tikzcd}$$

Proof. See [LMM+20, Lemma 2.7 and Lemma 2.8]. □

The Bousfield–Kuhn functor

Finally, we will need the Bousfield–Kuhn functor. The existence of this functor is an important and surprising structural feature of telescopic chromatic homotopy theory. It means that the $T(n)$-localisation of a spectrum X only depends on the underlying anima $\Omega^{\infty} X$.

Construction 7.1.27 (See for instance [Lur18a, Lec. 5 and 6])**.** Let V_n be a finite spectrum of type n and let $v_n \colon \Sigma^t V_n \to V_n$ denote a v_n-self map of V_n. Note that if we write $(\Phi_{V_n}(X))_0$ for the colimit

$$\Phi_{V_n}(X) = \mathrm{colim}\left(\mathrm{Map}_*(V_n, X) \xrightarrow{(v_n)^*} \mathrm{Map}_*(\Sigma^t V_n, X) \simeq \Sigma^{-t}\mathrm{Map}_*(V_n, X)\right),$$

we see that the self map v_n induces an equivalence $v_n^* \colon (\Phi_{V_n}(X))_0 \xrightarrow{\simeq} \Omega^t(\Phi_{V_n}(X))_0$. Hence, this assembles to an Ω–spectrum, and so we obtain a functor $\Phi_{V_n} \colon \mathrm{An}_* \to \mathrm{Sp}$.

By definition, we have that $v_n^{-1}\pi_*(X;V_n) \simeq \pi_*\Phi_{V_n}(X)$. In fact, it turns out (c.f. [Lur18a, Lec. 6]) by the Periodicity Theorem 7.1.13 that we can functorialise this construction across different choices of (V_n, v_n) to yield a functor

$$\Phi^{\geq n}\colon \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$$

determined as follows: for $E \in \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n}$, there exists some d and a type n pointed anima $V_n \in \mathrm{An}^{\omega}_*$ such that $\Sigma^\infty V_n \simeq \Sigma^d E$. Then $\Phi^{\geq n}(E) \simeq \Sigma^d \Phi_{V_n}$. Let

$$\Phi^{\geq n}_!\colon \mathrm{Sp} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$$

denote the left Kan extension of $\Phi^{\geq n}\colon \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n} \to \mathrm{Fun}(\mathrm{An}_*, \mathrm{Sp})$ along $\mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n} \hookrightarrow \mathrm{Sp}$.

Definition 7.1.28. Let $n \geq 1$. The Bousfield–Kuhn functor

$$\Phi_n\colon \mathrm{An}_* \to \mathrm{Sp}$$

is defined by $\Phi_n = \Phi^{\geq n}_!(\mathbb{S})$. Explicitly, we find that

$$\Phi_n(X) \simeq \lim_{\mathrm{Sp}_{\mathrm{ht}\geq n} \ni V \to \mathbb{S}} \Phi_V(X).$$

Theorem 7.1.29 (Bousfield–Kuhn, [Kuh08, Thm. 1.1])**.** *There is an equivalence $\Phi_n\Omega^\infty \simeq L_{T(n)}$, so we obtain a commutative diagram of categories*

$$\begin{tikzcd}
\mathrm{Sp} \arrow[r, "L_{T(n)}"] \arrow[d, "\Omega^\infty"'] & \mathrm{Sp}_{T(n)} \\
\mathrm{An}_* \arrow[ur, "\Phi_n"'] &
\end{tikzcd}$$

The following result is proved in [LMM+20, Proposition 2.9], where it is attributed to Gijs Heuts.

Proposition 7.1.30. *Let $f\colon X \to Y$ be a map of spectra and let $i \geq 1$. If $\Sigma^\infty\Omega^\infty f$ is a $T(i)$-local equivalence, then f is a $T(i)$-local equivalence.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\Omega^\infty X \arrow[r] \arrow[d] & \Omega^\infty\Sigma^\infty\Omega^\infty X \arrow[r] \arrow[d] & \Omega^\infty X \arrow[d] \\
\Omega^\infty Y \arrow[r] & \Omega^\infty\Sigma^\infty\Omega^\infty Y \arrow[r] & \Omega^\infty Y
\end{tikzcd}$$

where the horizontal rows are equivalences. Applying the Bousfield–Kuhn functor Φ_i we obtain a commutative diagram

$$\begin{tikzcd}
L_{T(i)}X \arrow[r] \arrow[d] & L_{T(i)}\Sigma^\infty\Omega^\infty X \arrow[r] \arrow[d, "\simeq"] & L_{T(i)}X \arrow[d] \\
L_{T(i)}Y \arrow[r] & L_{T(i)}\Sigma^\infty\Omega^\infty Y \arrow[r] & L_{T(i)}Y
\end{tikzcd}$$

where the horizontal composites are equivalences and the middle vertical map is an equivalence by assumption. This proves that $L_{T(i)}f$ is a retract of $L_{T(i)}\Sigma^\infty\Omega^\infty f$ which is an equivalence. ☐

Remark 7.1.31. The functor $\Sigma^\infty\Omega^\infty$ does not preserve $T(i)$-local equivalences. Indeed, the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{Z}$ is $T(i)$-acyclic since it is bounded above but $\Sigma^\infty\Omega^\infty\mathbb{Z}$ is a sum of spheres, thus not $T(i)$-acyclic.

The following ensures that $\Sigma^\infty\Omega^\infty$ preserves sufficiently connective $L_n^{p,f}$-equivalences:

Proposition 7.1.32 ([LMM+20, Prop. 2.11]). *Let $n \geq 1$ be an integer. Then there exists an integer $m \geq 2$ such that the following hold:*

1. *If F is an m-connective pointed anima such that $v_i^{-1}\pi_*(F;V_i) \simeq 0$ for $0 \leq i \leq n$, then $L_n^{p,f}\Sigma^\infty F \simeq 0$.*
2. *Let $f\colon X \to Y$ be an m-connective map of anima. If f is a v_i-periodic equivalence for $0 \leq i \leq n$ for each choice of basepoint, then $\Sigma^\infty f$ is an $L_n^{p,f}$-equivalence for each choice of basepoint.*
3. *The functor $\Sigma^\infty\Omega^\infty$ preserves m-connective $L_n^{p,f}$-equivalences.*

Proof. This relies on work of Bousfield. See [LMM+20, Proposition 2.11]. □

7.2 Telescopic purity of K–theory

Before we descend into the meat of the content, let us point out that, for $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, by virtue of Exercise 7.1.22 (2), we have the first equivalence in

$$L_{T(n)}\mathbb{K}(\mathcal{C}) \simeq L_{T(n)}\tau_{\geq 0}\mathbb{K}(\mathcal{C}) \simeq L_{T(n)}\mathrm{K}(\mathcal{C})$$

where the second equivalence is by Theorem 4.5.46. Hence, in this section we are free to pass between nonconnective and connective K–theory. It is often important to use $\mathbb{K}$ since this is the version of K–theory that is a localising invariant (it is even Karoubi–localising by Theorem 4.5.46).

To start with, let us note the following trick observation, which is apparently due to one of Mathew or Clausen.

Proposition 7.2.1 (Delooped +=Q, [LMM+20, Pf. of Lem. 3.6]). *Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$. Then $\Sigma\tau_{\geq 0}\mathbb{K}(\mathcal{C}) \simeq |\mathrm{k}(\mathrm{S}_\bullet(\mathcal{C})^{\simeq})| \in \mathrm{Sp}_{\geq 0}$.*

Proof. Recall by construction that we have an equivalence in CMon

$$\Omega^\infty\left(\Sigma\tau_{\geq 0}\mathbb{K}(\mathcal{C})\right) \simeq \Omega^\infty\left(\Sigma\tau_{\geq 0}\mathrm{K}(\mathcal{C})\right) \simeq |\mathrm{S}_\bullet(\mathcal{C})^{\simeq}|$$

where, as usual, the first equivalence is by construction of nonconnective K–theory Theorem 4.5.46 and since $\mathcal{C}$ was idempotent–complete. Hence, since the left hand term in this equivalence is grouplike and $|\mathrm{S}_\bullet(\mathcal{C})^{\simeq}|^{\mathrm{gp}} \simeq |(\mathrm{S}_\bullet(\mathcal{C})^{\simeq})^{\mathrm{gp}}| =: |\mathrm{k}(\mathrm{S}_\bullet(\mathcal{C})^{\simeq})|$, we get that $\Omega^\infty\left(\Sigma\tau_{\geq 0}\mathrm{K}(\mathcal{C})\right) \simeq |\mathrm{k}(\mathrm{S}_\bullet(\mathcal{C})^{\simeq})|$ and so an equivalence in the statement as required. □

Recall from Theorem 6.6.7 that for a connective ring spectrum A, $\Omega^\infty\tau_{\geq 1}\mathbb{K}(A)$ can be described as a plus–construction, so that there is a map

$$BGL(A) \longrightarrow \Omega^\infty\tau_{\geq 1}\mathbb{K}(A)$$

exhibiting the target as the plus–construction on the source.

Lemma 7.2.2 (Suspensions as plus–construction, [LMM+20, §2.3])**.** *Let A be a connective ring spectrum. Then the map $\Sigma^\infty BGL(A) \longrightarrow \Sigma^\infty\Omega^\infty\tau_{\geq 1}\mathbb{K}(A)$ is an equivalence.*

Proof. By Proposition 3.2.19 (1), we have an equivalence $\mathbb{S}[BGL(A)] \xrightarrow{\simeq} \mathbb{S}[\Omega^\infty\tau_{\geq 1}\mathbb{K}(A)]$. But since for pointed anima Z and pointed maps, we have a functorial splitting $\mathbb{S}[Z]\simeq\Sigma^\infty_+Z\simeq\Sigma^\infty Z\oplus\mathbb{S}$, we see that the map of interest is a retract of an equivalence, whence also an equivalence. □

Proposition 7.2.3 (Highly connective case, [LMM+20, Prop. 3.1])**.** *Let $n\geq 1$. There exists $N\geq 1$ such that the following holds: let $A\to B$ be an N–connective $L_n^{p,f}$–equivalence between connective ring spectra. Then the induced map $\mathrm{K}(A)\to\mathrm{K}(B)$ is again an $L_n^{p,f}$–equivalence.*

Proof. We take $N := m-1$ where m is the integer in Proposition 7.1.32. Since $A\to B$ is in particular a $T(0)=\mathbb{S}[p^{-1}]$–equivalence, by [LT19, Lem. 2.4] (which is just a slightly generalised version of Theorem 6.6.8 with essentially the same proof), we know that $\mathrm{K}(A)\to\mathrm{K}(B)$ is a $T(0)$–equivalence. Hence, it remains to prove that the map $\mathrm{K}(A)\to\mathrm{K}(B)$ is a $T(i)$–equivalence for $1\leq i\leq n$. By the general fact that $\tau_{\geq k}X\to X$ is a v_n–periodic equivalence for all k and $n\geq 1$ (cf. Exercise 7.1.22 (2)), it suffices to show that $\tau_{\geq 1}\mathrm{K}(A)\to\tau_{\geq 1}\mathrm{K}(B)$ is a $T(i)$–local equivalence for $1\leq i\leq n$.

We consider the commuting square

$$\begin{tikzcd}
\Sigma^\infty BGL(A) \arrow[r] \arrow[d, "\simeq"] & \Sigma^\infty BGL(B) \arrow[d, "\simeq"] \\
\Sigma^\infty\Omega^\infty\tau_{\geq 1}\mathrm{K}(A) \arrow[r] & \Sigma^\infty\Omega^\infty\tau_{\geq 1}\mathrm{K}(B)
\end{tikzcd}$$

where the equivalences are by virtue of Lemma 7.2.2. By Proposition 7.1.30 we need to show that the bottom horizontal map is a $T(i)$–equivalence for $1\leq i\leq n$, and so by the square above, it would suffice to show that the top horizontal map is a $T(i)$–equivalence. To this end, by virtue of Proposition 7.1.32, we would be done if we could show that the map $BGL(A)\to BGL(B)$ is m–connective and induces an isomorphism on v_i–periodic homotopy groups for $i\leq n$.

For this claim, recall the construction of GL and M from Construction 6.6.6 where $M(A) := \mathrm{colim}_r\,\Omega^\infty\mathrm{End}(A^r)\simeq\bigoplus^{\mathbb{N}}\Omega^\infty A$. As in the proof of Theorem 6.6.8, we know that $\mathrm{fib}(GL(A)\to GL(B))\simeq\mathrm{fib}(MA\to MB)$. Hence, $GL(A)\to GL(B)$ is $(m-1)$–connective whence $BGL(A)\to BGL(B)$ is m–connective.

Since $\Omega BGL(A) \simeq GL(A)$, we just need to show that $GL(A) \to GL(B)$ is a v_i–periodic equivalence for $1 \leq i \leq n$. By definition of GL, we have $\tau_{\geq 1} GL(A) \simeq \tau_{\geq 1} M(A)$. Moreover, since $A \to B$ is $L_n^{p,f}$–equivalence, it induces v_i–periodic equivalence for $i \leq n$. Hence, also $M(A) \to M(B)$ is a v_i–periodic equivalence, and finally by Exercise 7.1.22 (2) which implies that

$$T(i) \otimes GL(A) \simeq T(i) \otimes \tau_{\geq 1} GL(A) \simeq T(i) \otimes \tau_{\geq 1} MA \simeq T(i) \otimes MA,$$

also $GL(A) \to GL(B)$ is v_i–periodic equivalence for $i \leq n$, as desired. □

Proposition 7.2.4 (Truncating property of $T(n)$–local K–theory, [LMM+20, Prop. 3.4])**.** *For $n \geq 1$, $L_{T(n)}\mathrm{K}(-)$ is truncating on $L_n^{p,f}$–acyclic ring spectra.*

Proof. Letting N be the number provided in Proposition 7.2.3, the map $R \to \tau_{\leq N} R$ is an $L_{T(n)}\mathbb{K}$–equivalence by Proposition 7.2.3 for any $L_n^{p,f}$–acyclic ring spectrum R since the ring map $R \to \tau_{\leq N} R$ becomes the equivalence between the zero rings upon application of $L_n^{p,f}$. Now apply Lemma 6.8.7 to conclude. □

Corollary 7.2.5 ($T(i)$–vanishing of primary torsions, [LMM+20, Cor. 3.5])**.** *For any $r \geq 1$, $L_{T(i)}\mathrm{K}(\mathbb{Z}/p^r) \simeq 0$ for $i \geq 1$.*

Proof. Let $r \geq 1$ and $i \geq 1$. Since $L_{T(i)}\mathbb{K}$ is truncating on $L_i^{p,f}$-acyclic $\mathbb{E}_1$-rings by Proposition 7.2.4, we conclude in particular that $L_{T(i)}\mathbb{K}$ is nilinvariant on $L_i^{p,f}$-acyclic classical rings by Theorem 6.8.3 (or rather, a word–to–word modification of its proof but now just for $L_n^{p,f}$–acyclic classical rings). Since $\mathbb{Z}/p^r$ is bounded above, it follows that it is $T(k)$-acyclic for every $k \geq 1$ by Exercise 7.1.22 (1). Moreover, $\mathbb{Z}/p^r[p^{-1}] = 0$. Hence, $\mathbb{Z}/p^r$ is $L_i^{p,f}$-acyclic for any i. Hence, by nilinvariance, we see that

$$L_{T(i)}\mathrm{K}(\mathbb{Z}/p^r) \simeq L_{T(i)}\mathrm{K}(\mathbb{F}_p).$$

By Quillen's computation Theorem 7.3.4 and the +=S Theorem 5.3.17, we have that $\mathrm{K}(\mathbb{F}_p)^\wedge_p \simeq \mathrm{k}(\mathbb{F}_p)^\wedge_p \simeq \mathrm{H}\mathbb{Z}_p$ which is $T(i)$-acyclic since it is bounded above, again by Exercise 7.1.22 (1). Since the map $\mathrm{K}(\mathbb{F}_p) \to \mathrm{K}(\mathbb{F}_p)^\wedge_p$ is a $T(i)$-local equivalence by Exercise 7.1.22 (6), we conclude that $L_{T(i)}\mathrm{K}(\mathbb{Z}/p^r) \simeq L_{T(i)}\mathrm{K}(\mathbb{F}_p) \simeq 0$ as desired. □

We will unfortunately not prove the following "standard" result in this course since it will require a digression into weight structures. For the interested reader, we note that it is proved by a combination of the +=S Theorem 5.3.17 and the weighty theorem of the heart Theorem 5.1.5.

Theorem 7.2.6 ([LMM+20, §3.3])**.** *If R is a connective ring spectrum and $\mathrm{Proj}^\omega(R)$ is the category of finitely generated projective modules over R (i.e. the full subcategory of $\mathrm{Perf}(R)$ generated by R under finite direct sums and retracts), then we have $\mathrm{k}(\mathrm{Proj}^\omega(R)) \simeq \mathrm{K}(R)$.*

The next result illustrates a very cool manoeuvre of showing K–theoretic vanishing by showing vanishing at the level of endomorphism rings. In essence, this goes back to the categorification philosophy that K–theory is the study of rings via a multi–object version thereof, namely categories.

Proposition 7.2.7 ([LMM+20, Prop. 3.6])**.** *Let $\mathcal{C}$ be an additive category such that* $\mathrm{Map}_{\mathcal{C}}(X,X) \in \mathrm{CGrp} \simeq \mathrm{Sp}_{\geq 0}$ *is annihilated by* $L_n^{p,f}$ *for every* $X \in \mathcal{C}$*. Then:*

1. $L_{T(i)}\mathrm{k}(\mathcal{C}) \simeq 0$ *for* $1 \leq i \leq n$*,*
2. $L_{T(i)}\mathrm{K}(\mathcal{C}) \simeq 0$ *for* $1 \leq i \leq n$ *if* $\mathcal{C}$ *is stable.*

Proof. For (1), observe that $\mathcal{C}$ can be written as a filtered colimit of its full subcategories generated by finite direct sums and retracts by finitely many objects. Hence, passing to the direct sum of the finitely many generators and using that k commutes with filtered colimits, we may assume that $\mathcal{C}$ is generated under finite direct sums and retracts by a single object X. Hence, by an additive version of Schwede–Shipley, we get $\mathcal{C} \simeq \mathrm{Proj}^{\omega}(\mathrm{Map}_{\mathcal{C}}(X,X))$ the category of finitely generated projective modules over $\mathrm{Map}_{\mathcal{C}}(X,X) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{CGrp}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_{\geq 0})$, which is $L_n^{p,f}$–acyclic by hypothesis.

Now note that $\mathrm{K}(\mathrm{map}_{\mathcal{C}}(X,X))$ is $T(i)$–acyclic: this is because Proposition 7.2.4 says that $L_{T(i)}\mathrm{K}(\mathrm{map}_{\mathcal{C}}(X,X)) \simeq L_{T(i)}\mathrm{K}(\pi_0 \,\mathrm{map}_{\mathcal{C}}(X,X))$. On the other hand, as $\mathrm{map}_{\mathcal{C}}(X,X)$ is $L_n^{p,f}$–acyclic, in particular this means that $\pi_0 \,\mathrm{map}_{\mathcal{C}}(X,X)[p^{-1}] = 0$ and so the unit (and hence the whole associative unital ring) is p-power torsion, thus it is a module over $\mathbb{Z}/p^j$ for some large j. All in all, $L_{T(i)}\mathrm{K}(\mathrm{map}_{\mathcal{C}}(X,X))$ is a module over $L_{T(i)}\mathrm{K}(\mathbb{Z}/p^j)$ for some big j, and this is zero by Corollary 7.2.5. Then, since $\mathrm{k}(\mathcal{C}) \simeq \mathrm{K}(\mathrm{map}_{\mathcal{C}}(X,X))$ by Theorem 7.2.6, we conclude that $\mathrm{k}(\mathcal{C})$ is $T(i)$–acyclic for $1 \leq i \leq n$ as required.

For (2), recall from Proposition 7.2.1 that we have an equivalence in $\mathrm{Sp}_{\geq 0}$

$$\Sigma\mathrm{K}(\mathcal{C}) \simeq |\mathrm{k}(\mathrm{S}_\bullet(\mathcal{C})^{\simeq})|.$$

Now note that $\mathrm{Fun}(\Delta^j, \mathcal{C})$ still satisfies the mapping anima $L_n^{p,f}$–acyclicity condition: this is because for any two $X, Y \in \mathcal{C}$, the mapping spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$ is a left module over the $\mathbb{E}_1$–ring spectrum $\mathrm{map}_{\mathcal{C}}(Y,Y)$, which was assumed to be $L_n^{p,f}$–acyclic, whence the $L_n^{p,f}$–acylicity of $\mathrm{map}_{\mathcal{C}}(X,Y)$; now use that the mapping anima in $\mathrm{Fun}(\Delta^j, \mathcal{C})$ may be computed as a finite limit of mapping anima in $\mathcal{C}$, and so they must be $L_n^{p,f}$–acyclic also. Therefore, we can apply part (1) to get that the right hand side is $T(i)$–acyclic as required. □

Before proceeding further, let us record the following standard situation of a split Verdier sequence associated to idempotent algebras.

Construction 7.2.8 (Split Verdier sequences from idempotent algebras)**.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ be a presentably symmetric monoidal stable category, and so in particular it admits internal homs $\mathrm{Hom}_{\mathcal{C}}(-,-)$. Let $\mathbb{1}\to A$ be an idempotent algebra, and we write the associated fibre sequence as

$$I \longrightarrow \mathbb{1} \longrightarrow A.$$

Since $A\simeq A\otimes\mathbb{1}\to A\otimes A$ was assumed to be an equivalence, we obtain also that

$$A\otimes I\simeq 0 \qquad I\otimes I\xrightarrow{\simeq} I.$$

Now write

$$\mathcal{C}_A \coloneqq \big\{X\in\mathcal{C} \;:\; X\to X\otimes A \text{ is an equivalence }\big\}$$

$$\mathcal{C}_{I-\mathrm{tors}} \coloneqq \big\{X\in\mathcal{C} \;:\; I\otimes X\to X \text{ is an equivalence }\big\}.$$

Thus, by the relations above, we easily see that we have Verdier sequence

$$\mathcal{C}_A \xleftarrow{A\otimes -} \mathcal{C} \xleftarrow{\mathrm{incl}} \mathcal{C}_{I-\mathrm{tors}}.$$

Observe importantly that an object in $\mathcal{C}_A$ may also be characterised as those $X\in\mathcal{C}$ such that $\mathrm{Map}_{\mathcal{C}}(I\otimes Y, X)\simeq *$ for all $Y\in\mathcal{C}$.

Now, since $A\otimes -\colon \mathcal{C}\to\mathcal{C}_A$ is a smashing localisation (and in particular a Bousfield localisation with the inclusion as the right adjoint), by the yoga of split Verdier sequences [CDH+20, Lem. A.2.5], we see that the Verdier sequence above is right–split, i.e. the inclusion $\mathcal{C}_I\hookrightarrow\mathcal{C}$ admits a right adjoint also. Moreover, by the alternative characterisation of $\mathcal{C}_A$ above, it is easy to see that the inclusion $\mathcal{C}_A\subseteq\mathcal{C}$ also admits a further right adjoint given by the internal hom $\mathrm{Hom}_{\mathcal{C}}(A,-)$. Another application of [CDH+20, Lem. A.2.5] then yields the following split Verdier sequence

$$\begin{tikzcd}
\mathcal{C}_A \arrow[r, hook] & \mathcal{C} \arrow[l, bend right, "A\otimes -"'] \arrow[l, bend left, "\mathrm{Hom}_{\mathcal{C}}(A,-)"] \arrow[r, two heads, "L_I"] & \mathcal{C}_I \arrow[l, hook, bend right, "I\otimes -"'] \arrow[l, hook, bend left, "\mathrm{Hom}_{\mathcal{C}}(I,-)"]
\end{tikzcd}$$

where it follows formally that the right adjoint of L_I must be $\mathrm{Hom}_{\mathcal{C}}(I,-)$. Here, we set $\mathcal{C}_I\coloneqq\mathcal{C}\big[\{I\otimes X\to X\}^{-1}_{X\in\mathcal{C}}\big]$ and $L_I\colon\mathcal{C}\twoheadrightarrow\mathcal{C}_I$ is the associated Bousfield localisation, and we have used the identification $\mathcal{C}_I\simeq\mathcal{C}_{I-\mathrm{tors}}$. To see this, we just note that the functor $I\otimes -\colon\mathcal{C}\twoheadrightarrow\mathcal{C}_{I-\mathrm{tors}}$ satisfies the universal property of $\mathcal{C}_I$ since for any functor $\varphi\colon\mathcal{C}\to\mathcal{D}$ that inverts the maps $I\otimes X\to X$ must factor through $I\otimes -\colon\mathcal{C}\twoheadrightarrow\mathcal{C}_{I-\mathrm{tors}}$ since $\varphi(I\otimes X)\xrightarrow{\simeq}\varphi(X)$.

Note that in the situation above, while $\mathcal{C}_A$ sits in $\mathcal{C}$ in one way, $\mathcal{C}_I$ embeds into $\mathcal{C}$ in two distinct ways. Via the top inclusion $I\otimes -$, we are viewing $\mathcal{C}_I$ as the full subcategory $\mathcal{C}_{I-\mathrm{tors}}$ as defined above. On the other hand, via the bottom inclusion $\mathrm{Hom}_{\mathcal{C}}(I,-)$, we are viewing $\mathcal{C}_I$ as the full subcategory $\mathcal{C}_{I-\mathrm{cpl}}$ of I–complete objects consisting of those objects X such that $\mathrm{Map}_{\mathcal{C}}(A\otimes Y, X)\simeq *$ for all $Y\in\mathcal{C}$.

Exercise 7.2.9. Show that, under Construction 7.2.8, the fibre sequence

$$\Sigma^{-1}\mathbb{S}/p^{\infty} \longrightarrow \mathbb{S} \longrightarrow \mathbb{S}[p^{-1}]$$

from the proof of Lemma 7.1.25 gives rise to the p–local arithmetic fracture stable recollement.

Exercise 7.2.10. It is a fact that if $L\colon \mathcal{C} \rightleftharpoons \mathcal{D} : R$ is an adjunction between presentable categories where R itself admits a right adjoint, then for any $\mathcal{E} \in \mathrm{Pr}^L$, the given adjunction induces an adjunction

$$L \otimes \mathrm{id}_{\mathcal{E}}\colon \mathcal{C} \otimes \mathcal{E} \rightleftharpoons \mathcal{D} \otimes \mathcal{E} : R \otimes \mathrm{id}_{\mathcal{E}}.$$

Show using this fact that, if $L \dashv R$ was a Bousfield (co)localisation, then so is $L \otimes \mathrm{id}_{\mathcal{E}} \dashv R \otimes \mathrm{id}_{\mathcal{E}}$.

Notation 7.2.11. To avoid cluttering notation in the proof, we write $\mathcal{F}_{>n} \coloneqq \mathrm{Sp}^{\omega}_{\mathrm{ht}\geq n+1}$ for the category of p–local finite spectra of type $\geq n+1$, i.e. which are $K(0)\oplus\cdots\oplus K(n)$–acyclic.

Lemma 7.2.12 ([LMM+20, Lem. 3.7])**.** *For any ring spectrum A, there is a Verdier sequence*

$$\mathcal{F}_{>n} \otimes \mathrm{Perf}(A) \longrightarrow \mathrm{Perf}(A) \longrightarrow \mathrm{Perf}(L_n^{p,f}A)$$

and the endomorphism spectrum of every object in $\mathcal{F}_{>n} \otimes \mathrm{Perf}(A)$ is $L_n^{p,f}$–acyclic.

Proof. By Lemma 7.1.25 and Construction 7.2.8, we have a right split Verdier sequence

$$\mathrm{Ind}(\mathcal{F}_{>n}) \longrightarrow \mathrm{Sp} \longrightarrow \mathrm{Mod}_{\mathrm{Sp}}(L_n^{p,f}\mathbb{S})$$

all of whose right adjoints themselves admit further right adjoints. Hence, tensoring with $\mathrm{Mod}_{\mathrm{Sp}}(A)$ and using that $L_n^{p,f}$ is smashing, again from Lemma 7.1.25, we get by Exercise 7.2.10 the Verdier sequence

$$\mathrm{Ind}(\mathcal{F}_{>n}) \otimes \mathrm{Mod}_{\mathrm{Sp}}(A) \longrightarrow \mathrm{Mod}_{\mathrm{Sp}}(A) \longrightarrow \mathrm{Mod}_{\mathrm{Sp}}(L_n^{p,f}A).$$

Applying $(-)^{\omega}$ to this Verdier sequence and using the symmetric monoidal equivalence Theorem 2.2.7 then yields the required Verdier sequence.

Finally, $\mathcal{F}_{>n} \otimes \mathrm{Perf}(A)$ is generated by A–modules of the form $A \otimes F$ for $F \in \mathcal{F}_{>n}$, and these have endomorphism spectra $DF \otimes F \otimes A$, which are $L_n^{p,f}$–acyclic, and hence the endomorphism spectrum of every object is too. □

Exercise 7.2.13. Use [LT19, Lem. 2.4] to show that $\mathbb{K}(\mathbb{S}[p^{-1}]) \to \mathbb{K}(\mathbb{Z}[p^{-1}])$ is a p–local equivalence. **Hint:** *use that $\mathbb{S}[p^{-1}]_{(p)} \simeq \mathbb{Q} \simeq \mathbb{Z}[p^{-1}]_{(p)}$.*

With everything set up, we may now proceed to give the proof of the main theorem in [LMM+20].

Theorem 7.2.14 (LMMT purity, [LMM+20, Thm. 3.8])**.** *Let A be a ring spectrum. Then the map $A \to L_n^{p,f}A$ induces an equivalence on $L_{T(i)}\mathrm{K}(-)$ for $1 \leq i \leq n$. If $n \geq 2$, the map $A \to L_{T(1)\oplus\cdots\oplus T(n)}A$ induces an equivalence on $L_{T(n)}\mathrm{K}(-)$.*

Proof. By Lemma 7.2.12, we have

$$\mathrm{fib}(\mathbb{K}(A) \to \mathbb{K}(L_n^{p,f}A)) \simeq \mathbb{K}(\mathcal{F}_{>n} \otimes \mathrm{Perf}(A))$$

which is $T(i)$–acyclic for $1 \leq i \leq n$ by Lemma 7.2.12 and Proposition 7.2.7. For the second statement when $n \geq 2$, consider the pullback from Lemma 7.1.26

$$\begin{tikzcd}
L_n^{p,f}A \arrow[r] \arrow[d] & L_{T(1)\oplus\cdots\oplus T(n)}A \arrow[d] \\
A[\frac{1}{p}] \arrow[r] & (L_{T(1)\oplus\cdots\oplus T(n)}A)[\frac{1}{p}]
\end{tikzcd} \tag{7.1}$$

where we have used that $L_0^{p,f}X \simeq L_{T(0)}X \simeq X[\frac{1}{p}]$. Now, observe that

$$\mathcal{F}_{>0} \otimes \mathrm{Perf}(L_n^{p,f}A) \longrightarrow \mathcal{F}_{>0} \otimes \mathrm{Perf}(L_{T(1)\oplus\cdots\oplus T(n)}A)$$

is an equivalence. To see this, we need to show that for $M \in \mathcal{F}_{>0} \otimes \mathrm{Perf}(L_n^{p,f}A)$, the map $M \to L_{T(1)\oplus\cdots\oplus T(n)}M$ is an equivalence. As in the proof of Lemma 7.2.12, everything in $\mathcal{F}_{>0} \otimes \mathrm{Perf}(L_n^{p,f}A)$ is generated by $F \otimes L_n^{p,f}A$ for some $F \in \mathcal{F}_{>0}$. Hence, it is enough to show that $F \otimes L_n^{p,f}A \to F \otimes L_{T(1)\oplus\cdots\oplus T(n)}A$ is an equivalence (here we have commuted $F \otimes -$ with the localisations since this is just a finite colimit): this is an immediate consequence of (7.1) since $F[\frac{1}{p}] \simeq 0$. Therefore, all in all, together with Lemma 7.2.12, we get a pullback square of spectra

$$\begin{tikzcd}
\mathbb{K}(L_n^{p,f}A) \arrow[r] \arrow[d] & \mathbb{K}(L_{T(1)\oplus\cdots\oplus T(n)}A) \arrow[d] \\
\mathbb{K}(A[\frac{1}{p}]) \arrow[r] & \mathbb{K}((L_{T(1)\oplus\cdots\oplus T(n)}A)[\frac{1}{p}]).
\end{tikzcd}$$

By the first part of the theorem, it suffices to show that the top horizontal is a $T(i)$–equivalence for $i \geq 2$. Now the bottom terms are modules over $\mathbb{K}(\mathbb{S}[\frac{1}{p}])$ which is p–locally equivalent to $\mathbb{K}(\mathbb{Z}[\frac{1}{p}])$ by Exercise 7.2.13, and hence vanishes after $T(i)$–localisation for $i \geq 2$ by Mitchell's Theorem 7.3.6. □

This provides one half of the telescopic purity theorem. The other one is supplied by Clausen–Mathew–Naumann–Noel as a special of their more general [CMN+, Thm. C] regarding equivariant descent of chromatically localised K–theory. This provides the replacement for Mitchell's theorem in higher heights.

Theorem 7.2.15 (Clausen–Mathew–Naumann–Noel, [LMM+20, Thm. 1.1])**.** *For $n \geq 2$, $L_{T(n)}\mathrm{K}(L_{n-2}^{p,f}\mathbb{S}) \simeq 0$.*

Combining Theorem 7.2.14 and Theorem 7.2.15, we obtain the full purity result sought after.

Theorem 7.2.16 (Telescopic purity of K–theory, [LMM+20, Purity theorem])**.** *Let $n \geq 1$ and $A \to B$ be a map of $\mathbb{E}_1$–ring spectra which is a $T(n-1) \oplus T(n)$–equivalence. Then $K(A) \to K(B)$ is a $T(n)$–equivalence.*

Proof. The proof is very similar to the one of Theorem 7.2.14. First note that we have a pullback of rings

$$\begin{tikzcd}
L_n^{p,f} A \arrow[r] \arrow[d] & L_{T(n-1)\oplus T(n)} A \arrow[d] \\
L_{n-2}^{p,f} A \arrow[r] & L_{n-2}^{p,f} L_{T(n-1)\oplus T(n)} A
\end{tikzcd}$$

Instead of proceeding as in the aforementioned proof using elementary but tedious arguments via Lemma 7.2.12, we use now Theorem 6.6.9: since $L_{n-2}^{p,f}$ is smashing, the hypothesis of Theorem 6.6.9 is immediately satisfied, giving a pullback

$$\begin{tikzcd}
\mathbb{K}(L_n^{p,f} A) \arrow[r] \arrow[d] & \mathbb{K}(L_{T(n-1)\oplus T(n)} A) \arrow[d] \\
\mathbb{K}(L_{n-2}^{p,f} A) \arrow[r] & \mathbb{K}(L_{n-2}^{p,f} L_{T(n-1)\oplus T(n)} A).
\end{tikzcd}$$

By Theorem 7.2.14, it suffices to prove that the top horizontal is a $T(n)$–equivalence. Now each term in the bottom row is a module over $\mathrm{K}(L_{n-2}^{p,f}\mathbb{S})$, which vanishes $T(n)$–locally by Theorem 7.2.15, and so we are done. ☐

7.3 Redshift for $\mathbb{E}_\infty$-rings

The work of [LMM+20] forms a crucial part of the recent resolution of the redshift conjecture in algebraic K-theory for $\mathbb{E}_\infty$-rings. In this section, we wish to give an overview of this story.

Definition 7.3.1. An $\mathbb{E}_\infty$-ring R has height $n \geq 0$ if $L_{T(n)}R \not\simeq 0$ and $L_{T(n+1)}R \simeq 0$.

Remark 7.3.2. For an $\mathbb{E}_\infty$-ring R, we have that if $L_{T(i)}R \simeq 0$, then $L_{T(j)}R \simeq 0$ for $j > i$ by Hahn [Hah16]. In other words, we have that

$$\mathrm{supp}(R) = \{n \geq 0 \mid L_{T(n)}R \not\simeq 0\} = [0, n]$$

whenever R has height n. Compare this with Remark 7.1.6 where we see the opposite phenomena for finite spectra. It is also possible to define the height of an $\mathbb{E}_1$-ring using the notion of fp-type of Mahowald–Rezk [MR99] but it is more subtle since Hahn's vanishing result does not hold for $\mathbb{E}_1$-rings.

Example 7.3.3. We will say that $\mathbb{F}_p$ has height -1 since $\mathbb{F}_p[1/p] = 0$. Note that $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$, and $\mathbb{Z}_p$ have height 0. Furthermore, we have that $\mathrm{KU}, \mathrm{ku}, \mathrm{KO}$, and ko have height 1. The $\mathbb{E}_\infty$-ring tmf has height 2. In general, the nth Lubin–Tate spectrum E_n and the nth Morava K-theory $K(n)$ have height n.

We have the following examples:

Theorem 7.3.4 (Quillen)**.** $\mathrm{k}(\mathbb{F}_p)^\wedge_p \simeq \mathbb{Z}_p$.

Theorem 7.3.5 (Suslin)**.** $\mathrm{K}(\mathbb{C})^\wedge_p \simeq \mathrm{ku}^\wedge_p$.

Theorem 7.3.6 (Mitchell)**.** *If R is a classical commutative ring, then $L_{T(i)}\mathrm{K}(R) \simeq$ for $i \geq 2$.*

Based on empirical evidence, Ausoni–Rognes formulated their redshift philosophy:

Principle 7.3.7. If R is an $\mathbb{E}_\infty$-ring of height n, then $\mathrm{K}(R)$ has height $n+1$.

An equivalent way of phrasing Principle 7.3.7 is that $L_{T(n)}R \simeq 0$ if and only if $L_{T(n+1)}\mathrm{K}(R) \simeq 0$. The results stated above provide empirical evidence at low heights for this principle to be true. For instance, Mitchell's result verifies the redshift principle for $n = 1$ since discrete commutative rings are $T(1)$-acyclic.

Theorem 7.3.8 ([LMM+20; CMN+])**.** *Let R be an $\mathbb{E}_\infty$-ring. If $L_{T(n)}R \simeq 0$, then $L_{T(n+1)}\mathrm{K}(R) \simeq 0$. In particular, if R is of height n, then $\mathrm{K}(R)$ is of height $\leq n+1$.*

Proof. If $L_{T(n)}R \simeq 0$, then $L_{T(n+1)\oplus T(n)}R \simeq 0$ by the theorem of Hahn above, so it follows from Theorem 7.2.16 that

$$L_{T(n+1)}\mathrm{K}(R) \simeq L_{T(n+1)}\mathrm{K}(L_{T(n+1)\oplus T(n)}R) \simeq 0$$

which proves the desired claim. □

Theorem 7.3.9 ([Yua21])**.** *If E_n denotes a Lubin–Tate theory of height n, then*

$$L_{T(n+1)}\mathrm{K}(E_n) \not\simeq 0.$$

Remark 7.3.10. As a consequence of the work of Hahn–Wilson [HW22], we also find that there is an $\mathbb{E}_3$-BP-algebra form of $\mathrm{BP}\langle n\rangle$ such that $L_{T(n+1)}\mathrm{K}(\mathrm{BP}\langle n\rangle) \not\simeq 0$.

Theorem 7.3.9 proves that $\mathrm{K}(E_n)$ has height $\geq n+1$. Combining this with Theorem 7.3.8, we obtain that $\mathrm{K}(E_n)$ has height precisely $n+1$. This verifies Principle 7.3.7 for $R = E_n$.

Observation 7.3.11. If $R \to S$ is a map of $\mathbb{E}_\infty$-rings and $S \not\simeq 0$, then $R \not\simeq 0$. This is because a module over the zero ring is necessarily zero.

The redshift principle for $\mathbb{E}_\infty$-rings can therefore be verified by producing an $\mathbb{E}_\infty$-ring map into something of height n. This can be done by the following deep result of Burklund–Schlank–Yuan [BSY22, Corollary 5.2].

Theorem 7.3.12 ([BSY22])**.** *If R is a non-zero $T(n)$-local $\mathbb{E}_\infty$-ring, then there is a map of $T(n)$-local $\mathbb{E}_\infty$-rings $R \to E_n$, where E_n is a Lubin–Tate theory of height n.*

Corollary 7.3.13. *Let R be a non-zero $\mathbb{E}_\infty$-ring with* $\mathrm{height}(R) \geq 0$*. Then*

$$\mathrm{height}(\mathrm{K}(R)) = \mathrm{height}(R) + 1.$$

Proof. Assume that $\mathrm{height}(R) = n \geq 0$. By Theorem 7.3.12 there is a map of $\mathbb{E}_\infty$-rings

$$R \to E_n$$

for some Lubin–Tate theory of height n. This induces a map of $\mathbb{E}_\infty$-rings

$$\mathrm{K}(R) \to \mathrm{K}(E_n),$$

where the source has height $n+1$ by Theorem 7.3.9. It follows that $\mathrm{K}(R)$ has height $n+1$ by Observation 7.3.11. □

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