UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

Bachelor Thesis

GMTW

A Partial Exposition on the Galatius–Madsen–Tillman–Weiss Theorem

Søren Lund Skeie

Supervisor: Nathalie Wahl

10th January, 2025

GALATIUS–MADSEN–TILLMAN–WEISS

SØREN SKEIE

Abstract. In [3], Galatius, Madsen, Tillman, and Weiss determined the homotopy type of the classifying space of the embedded cobordism category. This result is now known as the Galatius–Madsen–Tillman–Weiss theorem or GMTW for short. We give a partial exposition on the proof of GMTW given in [4, §3], without tangential structures. The proof relies on certain moduli spaces of manifolds, defined using a family of sheaves $\Psi_{d,n}$ of topological spaces, which allow us to restate GMTW. Using these sheaves, the proof can be broken into three steps, we call: scanning, GMTW, and delooping. Unfortunately, due to lack of time, we have resorted to a rougher sketch, especially of the delooping argument, than the author would have wanted.

Contents

The author thanks Nathalie Wahl for help, guidance, and supervision. He also thanks Fadi Mezher and Søren Galatius for answering his small, technical, and sometimes stupid questions. It has been a wonderful experience working with a proof from *the book*.

Introduction

This thesis attempts to give a detailed exposition of some parts of a proof of the Galatius–Madsen–Tillman–Weiss theorem without tangential structures (GMTW) determining the homotopy type of the classifying space of the embedded cobordism category $\mathcal{C}_d$:

Theorem (GMTW)**.** *The spaces $B\mathcal{C}_d$ and $\Omega^{\infty-1}MT(d)$ are weakly equivalent.*

GMTW was originally proven in [3] by Søren Galatius, Ib Madsen, Ulrike Tillman, and Micheal Weiss from 2009, by constructing such a weak equivalence. This determines the homotopy type of $B\mathcal{C}_d$, as standard machinery computes the homotopy type of the space $\Omega^{\infty-1}MT(d)$.

Our exposition follows a different proof, given in [4, §3] by Søren Galatius and Oscar Randal-Williams from the following year of the, slightly stronger:

Main Theorem (4.2.1)**.** *There is a weak equivalence $B\mathcal{C}_d(\mathbb{R}^n) \simeq \Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp})$.*

We give a mathematical outline of the proof in the following section, which serves as our mathematical introduction to the thesis providing an overview of the main sections. Let us here account for the appendices: the proof we give, relies heavily on the use of certain moduli spaces, constructed using a family of sheaves $\Psi_{d,n}$. A detailed exposition on these sheaves is given in appendix C. For the most part, we will take the language of categories for granted, though the first two subsections of appendix A contains definitions from the less common and more topological dialect we use. The rest of appendix A defines $\mathbb{E}_n$-algebras, allowing us to address our shortcomings on the "delooping" argument of the proof. Appendices A and B, the former on categories and operads and the latter on sheaves, are barren deserts of definitions only meant for venturers with a specific goal in mind. Finally, appendix D contains some of the theory, the author of this thesis found most fun to re-formulate, from when he was trying to follow the original proof given in [3].

We would be amiss, if we did not mention that this thesis is also very inspired by unpublished lecture notes on diffeomorphism groups of manifolds by Alexander Kupers, and that applications of GMTW include: the Barratt–Priddy–Quillen–Segal theorem, Madsen and Weiss' generalized Mumford conjecture, as well as classifying (invertible) topological quantum field theories.

Notation & terminology. In this thesis, $A \subset B$ means $a \in A \Rightarrow a \in B$. We will use $\varnothing$ to denote $\emptyset$ when we think of $\emptyset$ as a manifold and remark that $\varnothing$ has *every* dimension.

Disclaimer. The author does not claim any originality in this project. He chose this topic for his thesis simply out of the desire to understand some of it. Hence the goal was never to attempt to write a "great" exposition on this proof of GMTW. There has also not been any attempt from the author to be historical. Being unaware of the proof of GMTW given in [4], he originally attempted to follow the proof given in [3]. It was not until 26$^{\text{th}}$ October, 2024 he first laid eyes on [4], which is a major reason this thesis only contains a *partial* exposition.

Outline

We begin with a sketchy outline of the proof of our main theorem, given in the following sections. For this, we must first introduce *cobordisms*:

Definition. Let $d \in \mathbb{N}$ and let M_0 and M_1 be closed $(d-1)$-dimensional smooth submanifolds of $\mathbb{R}^n$. A *d-dimensional cobordism W of length t* from M_0 to M_1 is a d-dimensional, compact, smooth submanifold of $[0,t] \times \mathbb{R}^n$ for some $t \in \mathbb{R}_{>0}$ satisfying

i) $W \cap (\{0\} \times \mathbb{R}^n) = \{0\} \times M_0$ and $W \cap (\{t\} \times \mathbb{R}^n) = \{t\} \times M_1$;
ii) $\partial W = \{0\} \times M_0 \cup \{t\} \times M_1$;
iii) there exists $\epsilon > 0$ such that

$$W \cap ([0,\epsilon) \times \mathbb{R}^n) = [0,\epsilon) \times M_0 \quad \text{and} \quad W \cap ((t-\epsilon, t] \times \mathbb{R}^n) = (t-\epsilon, t] \times M_1. \lrcorner$$

Classically, cobordisms can be used to define a coarse equivalence relation on manifolds: the existence of a d-dimensional cobordism between manifolds forms an equivalence relation on the set of closed $(d-1)$-dimensional smooth submanifolds of $\mathbb{R}^n$.

More importantly for this thesis, d-dimensional cobordisms form the morphisms of a category: We can form a category with objects the closed $(d-1)$-dimensional smooth submanifolds of $\mathbb{R}^n$, with non-identity morphisms the cobordisms between them as just defined, and with composition given by "concatenation" of cobordisms. In this thesis, we will work with a more sophisticated version of this cobordism category, which we denote $\mathcal{C}_d(\mathbb{R}^n)$. Our version is a topological category, and its full construction is given in construction 1.2.6.

The main theorem of this thesis, is that there is a weak equivalence

$$B\mathcal{C}_d(\mathbb{R}^n) \simeq \Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp}),$$

where $\operatorname{Th}(\gamma_{d,n}^{\perp})$ is the one-point compactification of the space of d-dimensional affine subspaces of $\mathbb{R}^n$. This equivalence is natural in n. There are standard techniques to compute the homotopy type of $\operatorname{Th}(\gamma_{d,n}^{\perp})$, so we will view this as determining the homotopy type of $B\mathcal{C}_d(\mathbb{R}^n)$.

Using naturality in n, GMTW follows by noting that

$$B\mathcal{C}_d = B\operatorname*{colim}_{n\to\infty}\mathcal{C}_d(\mathbb{R}^n) \cong \operatorname*{colim}_{n\to\infty} B\mathcal{C}_d(\mathbb{R}^n) \simeq \operatorname*{colim}_{n\to\infty}\Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp}) =: \Omega^{\infty-1}MT(d).$$

We will follow the proof given in [4, section 2–3], which uses moduli spaces. In our case, this means we will use spaces in which every point is a d-dimensional smooth manifold embedded in an open subset of euclidean space: For an open subset $U \subset \mathbb{R}^n$, we construct a space $\Psi_d(U)$ with underlying set the set of closed subsets of U which are smooth d-dimensional manifolds without boundary. The spaces $\Psi_d(U)$ have a rather technical-to-construct topology by Galatius–Randal-Williams, which the author has taken to calling the myopic topology. Due to the opacity of these technicalities we have moved them to an appendix, namely appendix C, and refer the thorough, or morbidly curious, reader to it. For now, we hope the more impatient reader will content themselves with the following overview of the most relevant qualities, for this thesis, of the topology, as well as how we will use it.

Relevant qualities of the myopic topology. As mentioned, the topologies assemble to a sheaf Ψ_d on $\mathbb{R}^n$. Aside from this, for a topological space X, continuity of a map $X \longrightarrow \Psi_d(U)$, is equivalent to continuity of the compositions

$$X \longrightarrow \Psi_d(U) \xrightarrow{\pi_K} (\Psi^{\mathrm{s}}|_U)_K$$

for every compact subset $K \subset U$, where $(\Psi^{\mathrm{s}}|_U)_K$ is the space of *germs near* K, i.e. two manifolds project to the same germ near K, if they agree on an open neighbourhood of K. In this sense, the myopic topology only *sees* what happens on compact subspaces; hence the name. We will also use theorem C.2.8 which states that the function

$$\mathrm{Emb}(U,V) \times \Psi_d(V) \longrightarrow \Psi_d(U) \quad \text{given by}$$
$$(j, M) \longmapsto j^{-1}(M)$$

is continuous.

An essential quality of the topology on $\Psi_d(U)$, is that it allows continuously pushing parts of a manifold to infinity, making them "disappear". A simple, illustrating example is the 0-dimensional manifold $\{t\} \subset \mathbb{R}$ for $t \in \mathbb{R}$, which, as $t \longrightarrow \infty$, converges to $\varnothing$ in $\Psi_0(\mathbb{R})$. Indeed, for every compact subset K of $\mathbb{R}$, the projection of this path is eventually constant, as the path leaves K.

The myopic topology plays an absolutely crucial rôle in the proof we present; It is what allows our geometric manipulations in each of the three major steps:

Scanning. Our first use of the myopic topology, after constructing $\mathcal{C}_d(\mathbb{R}^n)$, is to use a scanning argument to deform $\Psi_d(\mathbb{R}^n)$ to $\mathrm{Th}(\gamma_{d,n}^{\perp})$, essentially by stretching the manifolds to become affine d-planes or the point $\infty \in \mathrm{Th}(\gamma_{d,n}^{\perp})$.

Delooping. We let $\psi_d(n,k)$ denote the subspace of $\Psi_d(\mathbb{R}^n)$ consisting of those submanifolds M which satisfy $M \subset \mathbb{R}^k \times (-1,1)^{n-k}$. One can think of $\psi_d(n,k)$ as the subspace of those manifolds only allowed to be non-compact in the first k directions. We give $\psi_d(n,k)$ the base-point $\varnothing$ and identify $\psi_d(n,n) = \Psi_d(\mathbb{R}^n)$. As mentioned, the delooping argument will only be sketched, leaving no reason to also sketch it here. Instead we state the result of the argument: there is a weak equivalence $\psi_d(n,k) \simeq \Omega\psi_d(n,k+1)$ when $k > 0$.

GMTW. Scanning and delooping reduces our main theorem to showing that $B\mathcal{C}_d(\mathbb{R}^n)$ and $\psi_d(n,1)$ are weakly equivalent. To show this, we first construct another, simpler, topological category $D_d(\mathbb{R}^n)$ which models the cobordism category, in the sense that $B\mathcal{C}_d(\mathbb{R}^n)$ and $D_d(\mathbb{R}^n)$ are weakly equivalent. This $D_d(\mathbb{R}^n)$ is a topological poset, whose underlying set is the subset of $\mathbb{R} \times \psi_d(n,1)$ consisting of the pairs (t, M) satisfying that t is a regular value of the projection $x_1 \colon M \longrightarrow \mathbb{R}$ to the first coordinate, and whose partial order is defined by

$$(t, M) \leq (t, M') \quad \text{if and only if} \quad M = M' \text{ and } t \leq t'.$$

A morphism in $D_d(\mathbb{R}^n)$ therefore corresponds to a manifold M and an interval $[t_0, t_1]$. Showing that $B\mathcal{C}_d(\mathbb{R}^n) \simeq BD_d(\mathbb{R}^n)$ involves using the topology to note that the parts of the manifold whose first coordinate falls outside the interval are contractible datum, again essentially pushing everything outside this interval away to infinity.

Finally, unravelling the definition of $BD_d(\mathbb{R}^n)$ allows us to model $BD_d(\mathbb{R}^n)$ as a space in which a point is a manifold $M \in \psi_d(n,1)$ along with regular values $t_0 < t_1 < \ldots < t_k$ of $x_1 \colon M \longrightarrow \mathbb{R}$ each regular value having non-negative weights $a_0, \ldots, a_k \in \mathbb{R}_{\geq 0}$ such that $a_0 + \cdots + a_k = 1$, up to forgetting regular values with weight 0. With this model for $BD_d(\mathbb{R}^n)$, we have a forgetful map

$$u \colon BD_d(\mathbb{R}^n) \longrightarrow \psi_d(n,1),$$

defined by forgetting the regular values and their weights. We then end the proof by showing u is a weak equivalence.

Warning. From now on, we will take the construction of Ψ_d as a sheaf of topological spaces given in appendix C for granted.

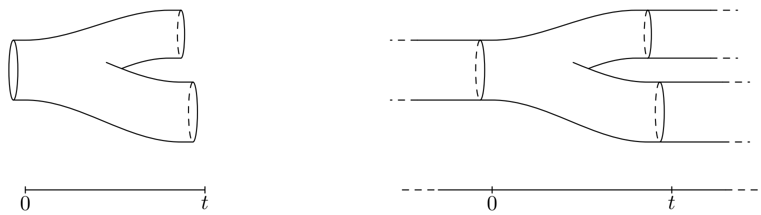
1. The cobordism category

1.1. Sketching the definition of the underlying category

Since the cobordism category is our main object of study in this thesis, we will attempt to give a detailed while understandable description of our main model of it. Our model is as a topological category, but to introduce it more gently, we will first sketch it as an ordinary category, in this section, and then give it a topology, in the next section.

We want $\mathcal{C}_d(\mathbb{R}^n)$ to be a topological model for the cobordism category sketched in the outline, namely a topological category with d-dimensional cobordisms as morphisms, closed $(d-1)$-dimensional manifolds embedded in $\mathbb{R}^n$ as objects, and concatenation as composition. The category $\mathcal{C}_d(\mathbb{R}^n)$ will be very similar to this category. The topology of $\mathcal{C}_d(\mathbb{R}^n)$ will come from the myopic topology on Ψ of Galatius–Randal-Williams, which does not consider manifolds with boundary. So, instead of usual cobordisms, as defined in the outline, we can record the length of each cobordism and then extend the cobordisms cylindrically at either end, obtaining a manifold without boundary (see fig. 1). Clearly, this does not add any information.

Figure 1. Extending cobordisms in the case of the classic "pair of pants" cobordism of length t (where $d = n = 2$, $M_0 = S^1$, and $M_1 = S^1 \sqcup S^1$) shown in fig. 1a to the manifold shown in fig. 1b.



(A) The ordinary cobordism embedded in $[0, t] \times \mathbb{R}^2$.

(B) The extended cobordism embedded in $\mathbb{R} \times \mathbb{R}^2$.

We can now define composition of these "extended" cobordisms as the extension of the concatenation of the "underlying" ordinary cobordisms giving us our cobordism category, where all manifolds involved are without boundary. Since all manifolds, in this new description, are without boundary, giving this category a topology is rather straight-forward using the myopic sheaf of Galatius–Randal-Williams.

1.2. As a topological category

Now, for the precise construction of $\mathcal{C}_d(\mathbb{R}^n)$ with its topology. We begin, by defining what will become the structure maps of $\mathcal{C}_d(\mathbb{R}^n)$:

Definition 1.2.1. Define the function $\sigma' \colon \Psi_{d-1}(\mathbb{R}^{n-1}) \longrightarrow \Psi_d(\mathbb{R}^n)$ by:

$$M \mapsto \mathbb{R} \times M. \qquad \lrcorner$$

Definition 1.2.2. Let $M \in \Psi_d(\mathbb{R}^n)$. If $a \in \mathbb{R}$ is a regular value of $x_1 \colon M \longrightarrow \mathbb{R}$, then $M_a := M \cap x_1^{-1}(a)$ is a smooth $(d-1)$-dimensional manifold. Thus we can define the partially defined function

$$\begin{aligned} \delta^a \colon \Psi_d(\mathbb{R}^n) &\dashrightarrow \Psi_{d-1}(\mathbb{R}^{n-1}) \quad \text{by} \\ M &\longmapsto M_a, \end{aligned}$$

defined for each $M \in \Psi_d(\mathbb{R}^n)$ for which a is a regular value of $x_1 \colon M \longrightarrow \mathbb{R}$. ⌟

Lemma 1.2.3. *The functions*

$$\sigma' \colon \Psi_{d-1}(\mathbb{R}^{n-1}) \longrightarrow \Psi_d(\mathbb{R}^n) \quad \textit{and} \quad \delta^a \colon \Psi_d(\mathbb{R}^n) \dashrightarrow \Psi_{d-1}(\mathbb{R}^{n-1})$$

are continuous (where defined) for all $a \in \mathbb{R}$.

Proof. Omitted due to lack of time. This is stated in definitions 3.1 and 3.3 of [4] with tangential structure and should be possible to check "by hand". □

Now, we introduce the spaces $\psi_d(n, k)$, which play a crucial rôle in this thesis.

Definition 1.2.4. Let $n, k \in \mathbb{N}_0$ such that $k \leq n$. Define $\psi_d(n, k)$ as the subspace of $\Psi_d(\mathbb{R}^n)$ consisting of those M satisfying $M \subset \mathbb{R}^k \times (-1, 1)^{n-k}$, where we identify $\mathbb{R}^0 \times (-1, 1)^n$ and $\mathbb{R}^n \times (-1, 1)^0$ with $(-1, 1)^n$ and $\mathbb{R}^n$ respectively. ⌟

One can think of $\psi_d(n, k)$ as the d-manifolds in $\mathbb{R}^n$ possibly non-compact in the first k directions.

Definition 1.2.5. We let

$$\sigma_k \colon \psi_{d-1}(n-1, k) \longrightarrow \psi_d(n, k+1)$$

denote the (co)restriction of σ'. We will not use σ' again in the rest of the thesis. ⌟

We will now give a formal construction of our topological category of d-dimensional cobordisms in $\mathbb{R}^n$:

Construction 1.2.6. We let $\mathcal{C}_d(\mathbb{R}^n)$ denote the topological category with space of objects and morphisms given by

$$\begin{aligned} \operatorname{ob} \mathcal{C}_d(\mathbb{R}^n) &= \psi_{d-1}(n-1, 0) \quad \text{and} \\ \operatorname{mor} \mathcal{C}_d(\mathbb{R}^n) &= \{0\} \times \operatorname{im} \sigma_0 \sqcup \mathscr{N}, \end{aligned}$$

where $\mathscr{N}$, the space of non-identity morphisms, is defined as the subspace

$$\mathscr{N} \subset (0, \infty) \times \psi_d(n, 1)$$

consisting of those (t, W) for which there exists $\epsilon > 0$ and $M', M'' \in \psi_{d-1}(n-1, 0)$ such that

$$\begin{aligned} W|_{(-\infty,\epsilon)\times\mathbb{R}^{n-1}} &= (\sigma_0(M'))|_{(-\infty,\epsilon)\times\mathbb{R}^{n-1}} \quad \text{and} \\ W|_{(t-\epsilon,\infty)\times\mathbb{R}^{n-1}} &= (\sigma_0(M''))|_{(t-\epsilon,\infty)\times\mathbb{R}^{n-1}}, \end{aligned}$$

so, in particular, 0 and t are regular values of $x_1 \colon W \longrightarrow \mathbb{R}$.

The structure maps of $\mathcal{C}_d(\mathbb{R}^n)$ are defined as follows:

$$\begin{aligned} s_0 \colon \psi_{d-1}(n-1, 0) &\xrightarrow{\sigma_0} \operatorname{im} \sigma_0 \cong \{0\} \times \operatorname{im} \sigma_0 \hookrightarrow \operatorname{mor} \mathcal{C}_d(\mathbb{R}^n), \\ d_1 \colon \{0\} \times \operatorname{im} \sigma_0 \sqcup \mathscr{N} &\longrightarrow \psi_{d-1}(n-1, 0), \end{aligned}$$

by $d_1|_{\{0\}\times\operatorname{im}\sigma_0}$ being the left-inverse of σ_0 and

$$d_1(t, W) = \delta^0(W), \quad \text{for all } (W, t) \in \mathscr{N},$$

and similarly, d_0 by $d_0|_{\{0\}\times\operatorname{im}\sigma_0}$ being the left-inverse of σ_0 and

$$d_0(t, W) = \delta^t(W), \quad \text{for all } (W, t) \in \mathscr{N}.$$

We define the composition by

$$(t, W) \circ (t', W') = (t + t', W''),$$

where W'' is the unique element of $\psi_d(n,1)$ such that

$$W''|_{(-\infty,t]\times\mathbb{R}^{n-1}} = W|_{(-\infty,t]\times\mathbb{R}^{n-1}} \quad \text{and}$$
$$W''|_{[t,\infty)\times\mathbb{R}^{n-1}} = (W' + te_1)|_{(-\infty,t]\times\mathbb{R}^{n-1}}.$$

⌟

Again, one should check that the composition is continuous, which we do not have time for.

2. Scanning

2.1. The homotopy type of the moduli space of d-manifolds

In this section we define the space $\mathrm{Th}(\gamma_{d,n}^{\perp})$ and construct a weak equivalence $\Psi_d(\mathbb{R}^n) \simeq \mathrm{Th}(\gamma_{d,n}^{n})$ by a scanning argument. There is standard theoretical machinery, which computes homotopical data about $\mathrm{Th}(\gamma_{d,n}^{\perp})$ and we will therefore consider this as "determining" the homotopy type of $\Psi_d(\mathbb{R}^n)$.

We begin with the definition of $\mathrm{Th}(\gamma_{d,n}^{\perp})$:

Definition 2.1.1. We let $\mathrm{Gr}_d(\mathbb{R}^n)$ denote the d-dimensional Grassmannian of $\mathbb{R}^n$ and let $\gamma_{d,n}^{\perp}$ denote the orthogonal complement of the tautological bundle over $\mathrm{Gr}_d(\mathbb{R}^n)$ i.e.: the $(n-d)$-dimensional vector bundle with base space $B(\gamma_{d,n}^{\perp}) = \mathrm{Gr}_d(\mathbb{R}^n)$, total space $E(\gamma_{d,n}^{\perp}) = \{(v, V) \in \mathbb{R}^n \times \mathrm{Gr}_d(\mathbb{R}^n) \mid v \in V^{\perp}\}$, and projection map

$$\begin{aligned} \pi(\gamma_{d,n}^{\perp})\colon E(\gamma_{d,n}^{\perp}) &\longrightarrow B(\gamma_{d,n}^{\perp}) \quad \text{defined by} \\ (v, V) &\longmapsto V, \end{aligned}$$

for all $(v, V) \in E(\gamma_{d,n}^{\perp})$. Finally, we define $\mathrm{Th}(\gamma_{d,n}^{\perp})$ as the one-point compactification of $E(\gamma_{d,n}^{\perp})$.[1] ⌟

We will follow the proof given in [2, §6] which works by constructing a map $q\colon \mathrm{Th}(\gamma_{d,n}^{\perp}) \longrightarrow \Psi_d(\mathbb{R}^n)$ and showing it is a weak equivalence.

Construction 2.1.2. Since $\mathrm{Th}(\gamma_{d,n}^{\perp})$ is the one-point compactification of $E(\gamma_{d,n}^{\perp})$, we will write $\mathrm{Th}(\gamma_{d,n}) = E(\gamma_{d,n}^{\perp}) \cup \{\infty\}$. Now, define $q\colon \mathrm{Th}(\gamma_{d,n}^{\perp}) \longrightarrow \Psi_d(\mathbb{R}^n)$ by letting

$$(v, V) \mapsto V + v$$

for all $(v, V) \in E(\gamma_{d,n}^{\perp})$ and $q(\infty) = \varnothing$. We will use that q restricts to an embedding of $E(\gamma_{d,n}^{\perp})$, which is stated on [2, p. 765]. That q is continuous at ∞ is a routine check. ⌟

To show q is a weak equivalence, we use the following homotopical lemma:

Lemma 2.1.3. *Let X be a topological space. If $U_0, U_1 \subset X$ form an open cover of X, then the pushout*

$$\begin{array}{ccc} U_0 \cap U_1 & \hookrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_1 & \hookrightarrow & X \end{array}$$

is a homotopy pushout.

Proof. This is lemma 32.1.5 of the lecture notes of Alexander Kupers on diffeomorphism groups of manifolds and a proof is given there. □

[1] The notation $\mathrm{Th}(\gamma_{d,n}^{\perp})$ is due to $\mathrm{Th}(\gamma_{d,n}^{\perp})$ being the *Thom space* of $\gamma_{d,n}^{\perp}$, but in the interest of brevity we will not elaborate on this further, though we, of course, encourage the reader to explore this, if they have not already.

We would have wanted to cover $\Psi_d(\mathbb{R}^n)$ by the subspaces $U_0, U_1 \subset \Psi_d(\mathbb{R}^n)$ where U_0 is the subspace of manifolds not containing the origin and U_1 is the subspace of those manifolds with a unique point closest to the origin. Unfortunately, in this case U_1 is not open. Instead, we tweak the definition of U_1 slightly:

Construction 2.1.4. We define $U_0 \subset \Psi_d(\mathbb{R}^n)$ as the set of those manifolds not containing the origin and we define U_1 as the subspace consisting of those $M \in \Psi_d(\mathbb{R}^n)$ such that the map

$$M \longrightarrow \mathbb{R} \quad \text{defined by}$$
$$x \longmapsto \|x\|^2 \quad \text{for all } x \in M$$

has a unique minimal value, which is the value of a non-degenerate critical point. ⌟

We will need the following technical lemma, whose proof is probably best skipped by the reader who has skipped appendix C:

Lemma 2.1.5. *The subspaces U_0 and U_1 of construction 2.1.4 form an open cover of $\Psi_d(\mathbb{R}^n)$.*

Partial proof. Due to lack of time, we will not show, that U_1 is open. To show $U_0 \cup U_1 = \Psi_d(\mathbb{R}^n)$ it suffices to show, that if $M \in \Psi_d(\mathbb{R}^n)$ contains 0, then 0 is a non-degenerate critical point of $\|-\|^2 \colon M \longrightarrow \mathbb{R}$. This follows from the inverse function theorem.

For each compact $K \subset \mathbb{R}^n$, let U_K denote the subset of $\Psi_d(\mathbb{R}^n)$ consisting of those manifolds that are disjoint from K and note that $U_0 = U_{\{0\}}$. We will show the more general result that U_K is open for all compact $K \subset \mathbb{R}^n$, because we will use it in a later proof: Let $K \subset \mathbb{R}^n$ be compact. Pick a manifold M disjoint from K and a neighbourhood A of M in $\Psi^{\mathrm{cs}}(\mathbb{R}^n)$ sufficiently small for every point of A to be disjoint from K. Now, since $\pi_K \colon \Psi^{\mathrm{cs}}(\mathbb{R}^n) \longrightarrow (\Psi^{\mathrm{s}}|_{\mathbb{R}^n})_K$ is open (lemma C.2.3) we get that $\pi_K(A) = \{\pi_K(\varnothing)\}$ is open and so $U_K = \pi_K^{-1}(\pi_K(\varnothing))$ is open in $\Psi_d^K(\mathbb{R}^n)$ and therefore also in $\Psi_d(\mathbb{R}^n)$.

For specific choices of M and A: use that K is compact and therefore bounded to choose a $k \in \mathbb{R}_{>0}$ such that $K \subset B_k$, where B_k denotes the ball centered at the origin with radius k. Now choose

$$M = S^d + (k+1) \cdot e_1 \quad \text{and} \quad A = c_M(\{s \in \Gamma_{\mathrm{c}}(\nu_M) \mid |s(x)| < 1\}). \qquad \square$$

The following theorem is [2, lemma 6.1].

Theorem 2.1.6. *The (co)restrictions*

$$q^{-1}(U_0) \longrightarrow U_0, \quad q^{-1}(U_1) \longrightarrow U_1, \quad \textit{and} \quad q^{-1}(U_0 \cap U_1) \longrightarrow U_0 \cap U_1$$

of q are weak equivalences. Consequently $q\colon \mathrm{Th}(\gamma_{d,n}^{\perp}) \longrightarrow \Psi_d(\mathbb{R}^n)$ is a weak equivalence.

Partial proof. This proof uses two homotopies H and S. Checking that these are continuous is a bit cumbersome and involves using the myopic topology. Therefore, we postpone showing H and S are continuous to the very end of the proof and, due to lack of time, we will not show that S is continuous when $t = 1$ though showing this only involves the defining property of smooth manifolds and the myopic topology.

First, we show why the last statement follows from the first: By lemma 2.1.3 and lemma 2.1.5 we have that

$$\begin{array}{ccc} q^{-1}(U_0 \cap U_1) & \hookrightarrow & q^{-1}(U_0) \\ \downarrow & & \downarrow \\ q^{-1}(U_1) & \hookrightarrow & \mathrm{Th}(\gamma_{d,n}^{\perp}) \end{array} \quad \text{and} \quad \begin{array}{ccc} U_0 \cap U_1 & \hookrightarrow & U_0 \\ \downarrow & & \downarrow \\ U_1 & \hookrightarrow & \Psi_d(\mathbb{R}^n) \end{array}$$

are homotopy pushouts, so, by homotopy invariance of the homotopy colimit, q must be weak equivalence.

We now show the mentioned (co)restrictions of q are weak equivalences: For a point $p \in \mathbb{R}^n$ and a real number $t \in \mathbb{R}$, let $\mu_{t,p}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denote the map defined by scalar multiplication by t at p i.e.:

$$\mu_{t,p}(x) = t \cdot (x - p) + p$$

for all $x \in \mathbb{R}^n$. Note that $\mu_{t,p}$ is a diffeomorphism when $t \neq 0$. We will let μ_t denote $\mu_{t,0}$ for all $t \in \mathbb{R}$.

First, we show that U_0 and $q^{-1}(U_0)$ are both contractible, implying the (co)restriction $q^{-1}(U_0) \longrightarrow U_0$ of q is a weak equivalence. We do this by giving contractions of U_0 to $\{\varnothing\}$ and of $q^{-1}(U_0)$ to $\{\infty\}$ by "pushing infinitely away": The homotopy

$$H\colon [0,1] \times U_0 \longrightarrow U_0 \quad \text{defined by}$$
$$(t, M) \longmapsto \begin{cases} \mu_{1-t}^{-1}(M) & t \neq 1 \\ \varnothing & t = 1 \end{cases}$$

is a contraction of U_0 to $\{\varnothing\}$.

Similarly, using that $q^{-1}(U_0) = \{(v, V) \in E(\gamma_{d,n}^{\perp}) \mid v \neq 0\} \cup \{\infty\}$, we get that the map

$$[0,1] \times q^{-1}(U_0) \longrightarrow q^{-1}(U_0), \quad \text{given by}$$
$$\infty \longmapsto \infty, \quad \text{and}$$
$$(v, V) \longmapsto \begin{cases} (\frac{1}{1-t} \cdot v, V) & t \neq 1 \\ \infty & t = 1, \end{cases} \quad \text{for all } (v, V) \in E(\gamma_{d,n}^{\perp})$$

is a contraction of $q^{-1}(U_0)$ to $\{\infty\}$.

Secondly, we define a deformation retraction of the (co)restriction $q^{-1}(U_1) \longrightarrow U_1$ of q: For each $M \in U_1$, let $p_M \in \mathbb{R}^n$ denote the unique point minimizing $\|x\|^2$ on M. Define

$$S\colon [0,1] \times U_1 \longrightarrow U_1 \quad \text{by}$$
$$(t, M) \longmapsto \begin{cases} (\mu_{1-t,p_M})^{-1}(M) & t \neq 1 \\ T_{p_M} M + p_M & t = 1, \end{cases}$$

for all $(t, M) \in [0,1] \times U_1$.

Thirdly, we note that the (co)restriction $[0,1] \times (U_0 \cap U_1) \longrightarrow U_0 \cap U_1$ of S is a deformation retraction of the (co)restriction $q^{-1}(U_0 \cap U_1) \longrightarrow U_0 \cap U_1$ of q.

Finally, we show that H and S are continuous beginning with H: Note that the function $\mu\colon [0,1) \longrightarrow \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ given by $t \mapsto \mu_{1-t}$ is continuous, so the restriction of H to $[0,1) \times U_0$ can be factored as the composition of the continuous maps

$$[0,1) \times U_0 \xrightarrow{\mu \times \mathrm{id}} \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \times U_0 \xrightarrow{p} U_0,$$

where p is the map from theorem C.2.8, so it is continuous. Checking, that H is continuous when $t = 1$ is equivalent to checking, that the composition

$$[0,1] \times U_0 \xrightarrow{H} U_0 \hookrightarrow \Psi_d(\mathbb{R}^n) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi_d^K(\mathbb{R}^n) \xrightarrow{\pi_K} (\Psi^{\mathrm{s}}|_{\mathbb{R}^n})_K \tag{2.1}$$

is continuous near $t = 1$ for all compact subsets $K \subset \mathbb{R}^n$. Let $K \subset \mathbb{R}^n$ be a compact subset and let $M \in U_0$. Use that K is compact and therefore bounded, to pick a $k \in \mathbb{R}_{>0}$ such that $K \subset B_k$, where B_k denotes the ball centered at the origin with radius k. Let $\epsilon = \mathrm{dist}(M, 0)$ (note that $\epsilon > 0$ since $M \in U_0$) and let A be the neighbourhood $([0,1] \cap (1 - \epsilon/2k, 1]) \times U_{B_{\epsilon/2}}$ of $(1, M)$ in $[0,1] \times U_0$, where $U_{B_{\epsilon/2}}$ is the neighbourhood of M in U_0 using the notation of the proof of lemma 2.1.5. It now suffices to check continuity on the neighbourhood A and here (2.1) is constantly $\pi_K(\varnothing)$ (because the minimal distance to the origin of any point in $H(A)$ is greater than k), finishing the proof that H is continuous.

Similarly for S: Note that the function $\alpha\colon U_1 \longrightarrow \mathbb{R}^n$ given by $M \mapsto p_M$ is continuous and that the function $\beta\colon \mathbb{R}^n \longrightarrow \mathrm{Diff}(\mathbb{R}^n)$ given by $x \mapsto \mathrm{tr}_x$, where $\mathrm{tr}_x\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ denotes translation by $x \in \mathbb{R}^n$, is continuous. Once we note that $\mu_{t,x} = \mathrm{tr}_x \circ \mu_t \circ \mathrm{tr}_x^{-1}$, it is clear that the restriction of S to $[0,1) \times U_1$ can be factored as the composition

$$\begin{aligned}
[0,1)\times U_1 &\xrightarrow{(\beta\circ\alpha\circ\mathrm{pr}_2,\mu\circ\mathrm{pr}_1,\beta\circ\alpha\circ\mathrm{pr}_2,\mathrm{pr}_2)} \mathrm{Diff}(\mathbb{R}^n)\times \mathrm{Diff}(\mathbb{R}^n)\times \mathrm{Diff}(\mathbb{R}^n)\times U_1\\
&\xrightarrow{\mathrm{id}\times\mathrm{id}\times\mathrm{inv}\times\mathrm{id}} \mathrm{Diff}(\mathbb{R}^n)\times \mathrm{Diff}(\mathbb{R}^n)\times \mathrm{Diff}(\mathbb{R}^n)\times U_1\\
&\xrightarrow{c\times\mathrm{id}} \mathrm{Diff}(\mathbb{R}^n)\times U_1\\
&\hookrightarrow \mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n)\times U_1\\
&\xrightarrow{p} U_1 \quad \text{(from theorem C.2.8)}
\end{aligned}$$

of continuous maps, where $\mathrm{inv}\colon \mathrm{Diff}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}(\mathbb{R}^n)$ is the map sending f to f^{-1} for all $f \in \mathrm{Diff}(\mathbb{R}^n)$ and $c\colon \mathrm{Diff}(\mathbb{R}^n)\times\mathrm{Diff}(\mathbb{R}^n)\times\mathrm{Diff}(\mathbb{R}^n) \longrightarrow \mathrm{Diff}(\mathbb{R}^n)$ is given by composition of functions i.e. $(x,y,z)\mapsto x\circ y\circ z$ (that inv and c are continuous are general facts of the Whitney C^∞-topology). Therefore, S is continuous on $[0,1)\times U_1$.

As mentioned, we omit the proof that S is continuous when $t=1$. We hope it is, at least intuitively, obvious to the reader. □

3. GMTW

3.1. The homotopy type of the cobordism category

In this section we will construct a weak equivalence $B\mathcal{C}_d(\mathbb{R}^n) \simeq \psi_d(n,1)$, where we take $B\mathcal{C}_d(\mathbb{R}^n)$ to mean $\left\| N^{\mathrm{int}}_\bullet \mathcal{C}_d(\mathbb{R}^n) \right\|$, as in definition A.2.2. We follow [4, §3.2] closely.

Remark. If X is a topological space and $\leq$ is a partial order on X, then the poset category of $(X, \leq)$ enhances (naturally) to a topological category, which we call a *topological poset*. In this case, we will abuse notation and also denote this topological category by X.

We begin by giving simpler models of the cobordism category as topological posets:

Definition 3.1.1. Let $D_d(\mathbb{R}^n) \subset \mathbb{R} \times \psi_d(n,1)$ be the subspace consisting of those $(t, M) \in \mathbb{R} \times \psi_d(n,1)$ for which t is a regular value of $x_1 \colon M \longrightarrow \mathbb{R}$. Give $D_d(\mathbb{R}^n)$ the partial order defined by

$$(t, M) \leq (t', M') \quad \text{if and only if} \quad M = M' \text{ and } t \leq t'.$$

Define $D_d^\perp(\mathbb{R}^n) \subset D_d(\mathbb{R}^n)$ as the subspace consisting of those $(t, M) \in D_d(\mathbb{R}^n)$ for which M is cylindrical in $x_1^{-1}(t-\epsilon, t+\epsilon)$ for some $\epsilon > 0$ and equip $D_d^\perp(\mathbb{R}^n)$ with the restriction of the partial order on $D_d(\mathbb{R}^n)$. ⌟

We will now show that $D_d(\mathbb{R}^n)$ and $D_d^\perp(\mathbb{R}^n)$ are models for our cobordism category, in the sense that their classifying spaces are homotopy equivalent to that of $\mathcal{C}_d(\mathbb{R}^n)$. To do this, we construct a continuous functor $c \colon D_d^\perp(\mathbb{R}^n) \longrightarrow \mathcal{C}_d(\mathbb{R}^n)$ and show that c and the inclusion $D_d^\perp(\mathbb{R}^n) \hookrightarrow D_d(\mathbb{R}^n)$ are level-wise homotopy equivalences of categories.

Construction 3.1.2. Let $c \colon D_d^\perp(\mathbb{R}^n) \longrightarrow \mathcal{C}_d(\mathbb{R}^n)$ be the continuous functor such that: On objects, c sends (t, M) to $\delta^t(M) = x_1^{-1}(t)$ and, on non-identity morphisms, c sends $(t_0 < t_1, M)$ to the extension of the "translated ordinary cobordism" $x_1^{-1}([t_0, t_1])$, more precisely: c sends $(t_0 < t_1, M)$ to the morphism from $\delta^{t_0}(M)$ to $\delta^{t_1}(M)$ given by the extension of $(M - t_0 e_1) \cap ([0, t_1 - t_0] \times \mathbb{R}^{n-1})$. ⌟

Lemma 3.1.3. *The zig-zag*

$$\mathcal{C}_d(\mathbb{R}^n) \xleftarrow{c} D_d^\perp(\mathbb{R}^n) \xhookrightarrow{\iota} D_d(\mathbb{R}^n),$$

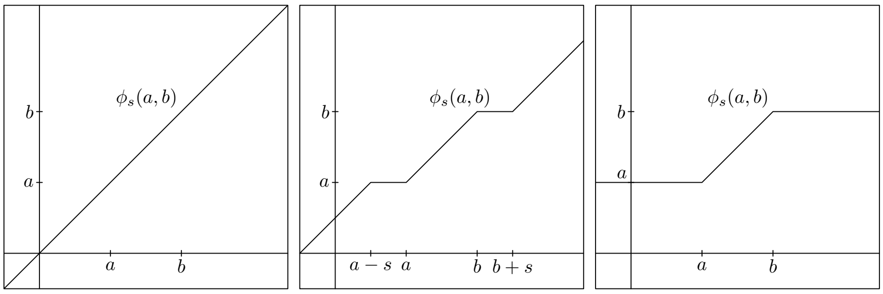
where c is the map from construction 3.1.2 and ι is the inclusion, is a zig-zag of level-wise homotopy equivalences of categories. In particular $B\mathcal{C}_d(\mathbb{R}^n) \simeq BD_d(\mathbb{R}^n)$.

Proof. That ι is a level-wise homotopy equivalence follows from [4, Lemma 3.4], which provides a homotopy essentially tweaking each manifold to be cylindrical near the specified regular values.

For a morphism $(t_0 < t_1, M)$ in $D_d^\perp(\mathbb{R}^n)$, c forgets the "ends" $x_1^{-1}((-\infty, t_0])$ and $x_1^{-1}([t_1, \infty))$ of M, and so to show that c is a level-wise homotopy equivalence of categories, we will have to show, that these form contractible data. To do this, we will use a family of maps $\phi_s(a,b)$ to continuously push either end away to infinity leaving us with our morphism of $\mathcal{C}_d(\mathbb{R}^n)$.

We introduce the functions $\phi_s(a,b)$ (see fig. 2):

Figure 2. Sketch of the graph of $\phi_s(a,b)$ as [4, Figure 1].



(A) case $s = 0$. (B) case $0 < s < \infty$. (C) case $s = \infty$.

For all $a, b \in \mathbb{R}$ with $a < b$ define $\phi_s(a,b)\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$x \xmapsto{\phi_s(a,b)} \begin{cases} x-(b-s) & x \geq b+s \\ b & b \leq x \leq b+s \\ x & a \leq x \leq b \\ a & a-s \leq x \leq a \\ x+(a-s) & a \leq a-s \end{cases}$$

for every $s \in \mathbb{R}_{\geq 0}$ and let $\phi_\infty(a,b)$ denote the point-wise limit of $\psi_s(a,b)$ as $s \to \infty$. Note that $\phi_0(a,b) = \mathrm{id}_{\mathbb{R}}$ whenever defined and that for a morphism $(t_0 < t_1, M)$ in $D_d^{\perp}(\mathbb{R}^n)$, we have that

$$(\phi_s(a,b) \times \mathrm{id}_{\mathbb{R}^{n-1}})^{-1}(M)$$

is an element of $\psi_d(n,1)$ which agrees with M on $[t_0,t_1] \times \mathbb{R}^{n-1}$, is cylincrical on $[t_0-s,t_0]\times\mathbb{R}^{n-1}$ and $[t_1,t_1+s]\times\mathbb{R}^{n-1}$, and is a translated copy of $x_1^{-1}((-\infty,t_0])$ and $x_1^{-1}([t_1,\infty))$ on $(-\infty,t_0-s]\times\mathbb{R}^{n-1}$ and $[t_1+s,\infty)\times\mathbb{R}^{n-1}$ respectively. Therefore, we can (re)write

$$c(t_0 < t_1, M) = (t_1 - t_0, (\phi_\infty(t_0,t_1) \times \mathrm{id}_{\mathbb{R}^{n-1}})^{-1}(M) - t_0 \cdot e_1).$$

We will now show, that c is a level-wise homotopy equivalence of categories. Let $k \in \mathbb{N}_0$. We show, that the "inclusion" $N_k^{\mathrm{int}}\mathcal{C}_d(\mathbb{R}^n) \hookrightarrow N_k^{\mathrm{int}}D_d^{\perp}(\mathbb{R}^n)$ is a homotopy inverse of $N_k^{\mathrm{int}}c$. On the image of this inclusion, $N_k^{\mathrm{int}}c$ restricts to the identity, so we only need to provide a homotopy from the identity on $N_k^{\mathrm{int}}D_d^{\perp}(\mathbb{R}^n)$ to c post-composed with the inclusion:

Define the homotopy $h\colon [0,1] \times N_k^{\mathrm{int}}D_d^{\perp}(\mathbb{R}^n) \longrightarrow N_k^{\mathrm{int}}D_d^{\perp}(\mathbb{R}^n)$ by letting $h(s,-)$ send the chain $(t_0 < \ldots < t_k, M)$ in $N_k^{\mathrm{int}}D_d^{\perp}(\mathbb{R}^n)$ to the chain

$$(t_0 - s\cdot t_0 < \ldots < t_k - s \cdot t_0, (\phi_{\frac{s}{1-s}}(t_0,t_k) \times \mathrm{id}_{\mathbb{R}^{n-1}})^{-1}(M) - s \cdot t_0 \cdot e_1),$$

for all $s \in [0,1]$. This homotopy suffices, finishing the proof. Checking continuity of h is similar to checking continuity of the homotopies in the proof of theorem 2.1.6 and is omitted due to lack of time. □

Construction 3.1.4. A chain of m composeable morphisms in $D_d(\mathbb{R}^n)$ is of the form $(t_0 \leq \ldots \leq t_m, M)$ where $M \in \psi_d(n,1)$ and where t_i is a regular value of $x_1\colon M \longrightarrow \mathbb{R}$ for each $i = 0, \ldots, m$. Writing out the canonical models for

$$BD_d(\mathbb{R}^n) = \left\| N_\bullet^{\mathrm{int}} D_d(\mathbb{R}^n) \right\| = \int^{[m]\in\mathbf{\Delta}_+} N_m^{\mathrm{int}} D_d(\mathbb{R}^n) \times \Delta^m_{\mathbf{Top}},$$

we can therefore find the following model for $BD_d(\mathbb{R}^n)$:

We give each point of $BD_d(\mathbb{R}^n)$ coordinates which we will write as

$$[a_0\bar{t}_0 + \ldots + a_n\bar{t}_m; M],$$

where $M \in \psi_d(n,1)$ and t_i is a regular value of $x_1\colon M \longrightarrow \mathbb{R}$ for each $i = 0, \ldots, m$, and where $(a_0, \ldots, a_m)$ is a point of the standard m-simplex in $\mathbb{R}^{m+1}$ (thought of as having vertices labeled by $\bar{t}_0, \ldots, \bar{t}_m$).[2] Furthermore, we call a_i the *weight* of t_i for each $i = 0, \ldots, m$, and impose that removing a regular value with weight 0 gives coordinates of the same point, so e.g.:

$$[0\bar{t}_0 + a_1\bar{t}_1 + \ldots + a_m\bar{t}_m; M] = [a_1\bar{t}_1 + \ldots + a_m\bar{t}_m; M].$$

From this description, the topology on $BD_d(\mathbb{R}^n)$ should be clear.

With these coordinates, we now define the forgetful map

$$u\colon BD_d(\mathbb{R}^n) \longrightarrow \psi_d(n,1) \quad \text{defined on all points by}$$
$$[a_0\bar{t}_0 + \ldots + a_m\bar{t}_m; M] \longmapsto M.$$ ⌟

We now want to show that u is a weak equivalence. To do this, we will use the following homotopical lemma.

Lemma 3.1.5. *Let $e\colon X \longrightarrow Y$ be a map of topological spaces. The map e is a weak equivalence if and only if each commutative diagram of the form*

$$\begin{array}{ccc} \partial D^n & \xrightarrow{\hat{f}} & X \\ \big\downarrow & & \big\downarrow e \\ D^n & \xrightarrow[f]{} & Y \end{array}$$

admits a lift $g\colon D^n \longrightarrow X$ making the lower right triangle and the upper left triangle of

$$\begin{array}{ccc} \partial D^n & \xrightarrow{\hat{f}} & X \\ \big\downarrow & \overset{g}{\nearrow} & \big\downarrow e \\ D^n & \xrightarrow[f]{} & Y \end{array}$$

strictly commute and commute up to homotopy respectively (for all $n \in \mathbb{N}_0$).

Proof. We omit the proof, but note that it follows from [7, §9.6 Lemma on p. 68]. □

Finally, we show that u is a weak equivalence:

Theorem 3.1.6. *The map $u\colon BD_d(\mathbb{R}^n) \longrightarrow \psi_d(n,1)$ from construction 3.1.4 is a weak equivalence.*

Proof. We will use lemma 3.1.5. Let f and $\hat{f}$ be continuous maps such, that

$$\begin{array}{ccc} \partial D^m & \xrightarrow{\hat{f}} & BD_d(\mathbb{R}^n) \\ \big\downarrow & & \big\downarrow u \\ D^m & \xrightarrow[f]{} & \psi_d(n,1) \end{array}$$

commutes for $m \in \mathbb{N}_0$. We will now construct a continuous map $g\colon D^m \longrightarrow BD_d(\mathbb{R}^n)$ as in lemma 3.1.5:

For every $t \in \mathbb{R}$ let U_t denote the set of points $x \in D^m$ such that t is a regular value of $x_1\colon f(x) \longrightarrow \mathbb{R}$. Note that U_t is open for every $t \in \mathbb{R}$. By Sard's theorem, the collection $\{U_t\}_{t\in\mathbb{R}}$ is an open cover of D^m. Since D^m is compact there exists

[2]The notation $\bar{t}_i$ is only meant to distinguish the label of the vertex in the simplex from the real number t_i. A way to make $a_0\bar{t}_0 + \ldots + a_m\bar{t}_m$ formal is as an element of the free $\mathbb{R}$-module on the set of regular values of $x_1\colon M \longrightarrow \mathbb{R}$, incidentally satisfying $a_0 + \cdots + a_m = 1$.

finitely many $t_0 < \ldots < t_k \in \mathbb{R}$ such that $\{U_{t_i}\}_{i\in[k]}$ is an open cover of D^m. Pick a partition of unity $a_0, \ldots, a_k \colon D^m \longrightarrow [0,1]$ subordinate to $\{U_{t_i}\}_{i\in[k]}$. Now define $g\colon D^m \longrightarrow BD_d(\mathbb{R}^n)$ by

$$x \overset{g}{\mapsto} [a_0(x)\bar{t}_0 + \ldots + a_k(x)\bar{t}_k; f(x)]$$

for all $x \in D^m$, where we do not add the $a_i(x)\bar{t}_i$ if $a_i(x) = 0$ to make it well-defined.

Now, clearly $u \circ g = f$. Since $u \circ g = f$ we have $u \circ \hat{f} = u \circ g|_{\partial D^m}$. The fibre of u over a point $M \in \psi_d(n,1)$ is a simplex whose vertices correspond to regular values of $x_1 \colon M \longrightarrow \mathbb{R}$. So since simplices are convex we can pick the straight-line homotopy on the fibres of u as a homotopy between $\hat{f}$ and $g|_{D^m}$. □

4. Delooping

4.1. Sketching the delooping argument

As mentioned in the outline and introduction, this section has, unfortunately, due to lack of time, only managed to become a *sketch* of the delooping argument needed to finish the proof. To finish the proof, we need that $\psi_d(n,1) \simeq \Omega^{n-1}\Psi_d(\mathbb{R}^n)$, and, to prove this, it suffices that $\psi_d(n,k) \simeq \Omega\psi_d(n,k+1)$, when $k > 0$.

We begin with a quick argument that $\psi_d(n,1)$, abstractly, is an $(n-1)$-fold loop space, i.e. it does not tell us *which* space $\psi_d(n,1)$ is an $(n-1)$-fold loop space of, and so does not quite suffice for the proof of GMTW:

It is fairly elementary to show, using theorem C.2.8, that $\psi_d(n,1)$ has the structure of an algebra over the little $(n-1)$-cubes operad $\mathcal{D}_{n-1}$, making it an $\mathbb{E}_{n-1}$-algebra, as defined in definition A.5.4. As we will note later, [4] shows that $\psi_d(n,1)$ is group-like with respect to this structure and so it follows from the *May recognition principle*, [11, Theorem 1.1.6], that $\psi_d(n,1)$ is an $(n-1)$-fold loop-space.

Before we give an outline of how [4] proves that $\psi_d(n,k) \simeq \Omega\psi_d(n,k+1)$ when $k > 0$, we will mention that Oscar Randal-Williams gives another proof, in higher generality, of this in [9, §5–6]. The author of this thesis does not claim to understand this proof, but to the best of his knowledge the following should be a true statement about it: Randal-Williams uses a more general definition of $\Psi_{d,n}$, which he shows is microflexible and then uses Gromov's h-principle to conclude a statement from which it follows that $\psi_d(n,k) \simeq \Omega\psi_d(n,k+1)$ when $k > 0$.

Let us now give the same appetizer as [4], before we outline how they actually prove it, as the prove they give is simpler than that of the appetizer:

Appetizer. The map

$$\begin{aligned} \mathbb{R} \times \psi_d(n,k) &\longrightarrow \psi_d(n,k+1) \\ (t, M) &\longmapsto M + t \cdot e_{k+1} \end{aligned}$$

extends uniquely to a map $S^1 \wedge \psi_d(n,k) \longrightarrow \psi_d(n,k+1)$ and the adjunct

$$\psi_d(n,k) \longrightarrow \Omega\psi_d(n,k+1)$$

of this map is a homotopy equivalence.

Outline of delooping argument. Now, we will outline how [4] proves

$$\psi_d(n,k) \simeq \Omega\psi_d(n,k+1) \quad \text{when } k > 0.$$

The key lemma, [4] makes use of, is a well-know result of Segal. To state it, we use the following definition:

Definition 4.1.1. A *strong Segal space* is a simplicial space for which the Segal maps induce homotopy equivalences. Let $X_\bullet$ be a strong Segal space, then the Segal maps induce a monoid structure on $\pi_0 X_1$. If the induced monoid structure on $\pi_0 X_1$ forms a group, then we say $X_\bullet$ is *group-like.* ⌟

As promised, the aforementioned result of Segal:

Lemma 4.1.2. *Let $X_\bullet$ be a strong Segal space. Then the natural map*

$$X_1 \longrightarrow \Omega\|X_\bullet\|$$

is a homotopy equivalence if and only if $X_\bullet$ is group-like.

To use this result, they endow $\pi_0\psi_d(n,k)$ with a monoidal structure when $k<n$ by "stacking" the manifolds on top of each other in the $(k+1)^{\mathrm{st}}$ direction and squeezing them to fit in the $(k+1)^{\mathrm{st}}$ direction. Formally: we define the sum

$$\oplus_k\colon \psi_d(n,k)\times\psi_d(n,k)\longrightarrow\psi_d(n,k),\quad \text{by}$$
$$M_1\oplus_k M_2 := (\mu_{k+1,2}^{-1}(M_1)-1/2\cdot e_{k+1})\cup(\mu_{k+1,2}^{-1}(M_1)+1/2\cdot e_{k+1})$$

for all $M_1, M_2\in\psi_d(n,k)$, where $\mu_{k+1,2}\colon\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is the smooth embedding which scales the $(k+1)^{\mathrm{st}}$ coordinate by 2. The monoid structure on $\pi_0\psi_d(n,k)$ is then defined by using $\oplus_k$ on representatives.[3]

To apply lemma 4.1.2, [4] constructs a strong Segal space $\mathcal{N}_\bullet\psi_d(n,k)$, which plays the role of the nerve of the monoid, in the sense that there is a natural homotopy equivalence

$$\psi_d(n,k)\simeq\mathcal{N}_1\psi_d(n,k)\qquad\text{(this is [4, Prop. 3.19])}$$

and the monoid structure induced on $\pi_0\psi_d(n,k)$ agrees with the one we defined previously, while being easier to work with for our purposes. The construction of $\mathcal{N}_\bullet\psi_d(n,k)$ is not difficult, but it is long and a bit notationally heavy, and their proof that $\mathcal{N}_\bullet\psi_d(n,k)$ is a strong Segal space ([4, Lemma 3.18]) is quite similar to the proof of lemma 3.1.3.

Proposition 3.6 of [4] shows that σ_k of definition 1.2.5 induces a bijection

$$\pi_0\sigma_k\colon\pi_0\psi_{d-1}(n-1,k)\longrightarrow\pi_0\psi_d(n,k+1),$$

and, since $\sigma_k(M_0\oplus_k M_1)=\sigma_k(M_0)\oplus_{k+1}\sigma_k(M_1)$, this bijection respects the monoid structure. Therefore, we get monoid isomorphisms

$$\pi_0\psi_d(n,k)\cong\pi_0\psi_{d-k+1}(n-k+1,1),$$

so to show that $\mathcal{N}_\bullet\psi_d(n,k)$ is group-like, it suffices to show that $\pi_0\psi_d(n,1)$ is a group. That $\pi_0\psi_d(n,1)$ is a group, follows from Corollary 3.11 of [4], which states that it is isomorphic to the *bordism monoid* $\Omega_{d,n}$ of cobordism classes of $(d-1)$-dimensional manifolds in $\mathbb{R}^{n-1}$, which is well-known to be a group.

By lemma 4.1.2, the problem is now reduced to providing a weak equivalence

$$\|\mathcal{N}_\bullet\psi_d(n,k)\|\longrightarrow\psi_d^{\varnothing}(n,k+1),$$

where $\psi_d^{\varnothing}(n,k+1)$ denotes the path component of $\varnothing$ of $\psi_d(n,k+1)$, because then, taking loops, we get a weak equivalence

$$\psi_d(n,k)\simeq\Omega\|\mathcal{N}_\bullet\psi_d(n,k)\|\longrightarrow\Omega\psi_d^{\varnothing}(n,k+1)=\Omega\psi_d(n,k+1).$$

This is done by a combination of the propositions 3.20 and 3.31 of [4]. The proofs of both of these, the former being quite short and the latter quite long, use similar methods to the ones we have used throughout this thesis, especially in section 3. This concludes our sketch of the argument of the following theorem:

Theorem 4.1.3. *There is a weak equivalence $\psi_d(n,k)\simeq\Omega\psi_d(n,k+1)$ when $k>0$.*

4.2. Concluding GMTW

Our previous work amounts to a proof of our main theorem:

Theorem 4.2.1. *There is a weak equivalence $B\mathcal{C}_d(\mathbb{R}^n)\simeq\Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp})$.*

Proof. We combine the weak/homotopy equivalences of lemma 3.1.3, theorem 3.1.6, theorem 4.1.3, and theorem 2.1.6 to give the weak equivalence

$$B\mathcal{C}_d(\mathbb{R}^n)\simeq BD_d(\mathbb{R}^n)\simeq\psi_d(n,1)\simeq\Omega^{n-1}\Psi_d(\mathbb{R}^n)\simeq\Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp}).\qquad\square$$

[3] So, we have $n-k$ different monoidal structures on $\pi_0\psi(n,k)$ and the $\mathbb{E}_{n-1}$-algebra structure on $\psi_d(n,1)$ is the one inducing each of these, when $k=1$.

When taking for granted, that the weak equivalence given in the proof of theorem 4.2.1 is natural in n, we get the main theorem of [3] without tangential structures as a corollary:

Corollary (GMTW)**.** *There is a weak equivalence* $B\mathcal{C}_d \simeq \Omega^{\infty-1}MT(d)$.

Proof. Taking the colimit as $n \longrightarrow \infty$, we get

$$B\mathcal{C}_d = B \operatorname*{colim}_{n\to\infty} \mathcal{C}_d(\mathbb{R}^n) \cong \operatorname*{colim}_{n\to\infty} B\mathcal{C}_d(\mathbb{R}^n) \simeq \operatorname*{colim}_{n\to\infty} \Omega^{n-1}\operatorname{Th}(\gamma_{d,n}^{\perp}) =: \Omega^{\infty-1}MT(d).$$

□

Appendix A. Categories & Operads

This appendix' *raison d'être* is simply to contain definitions used in other sections and appendices. A reader who is actually interested in learning these definitions should figure out, which diagrams are required to commute in each case themselves and only use this as a way to check their solutions. We consider the theory of this appendix well-known.

A.1. Internal categories

Definition A.1.1. Let $\mathscr{C}$ be a category. A *category C internal to $\mathscr{C}$* consists of objects $C_0, C_1 \in \mathscr{C}$ and morphisms

$$C_0 \underset{d_0}{\overset{d_1}{\leftleftarrows}}\!\!\xrightarrow{s_0} C_1 \xleftarrow{\;c\;} C_1 \times_{C_0} C_1,$$

where $C_1 \times_{C_0} C_1$ is the pullback

$$\begin{CD} C_1 \times_{C_0} C_1 @>{p_2}>> C_1 \\ @V{p_1}VV @VV{d_1}V \\ C_1 @>>{d_0}> C_0, \end{CD}$$

such that the diagrams

$$C_0 \xrightarrow{s_0} C_1, \quad d_1 \circ s_0 = \mathrm{id}_{C_0} = d_0 \circ s_0 \colon C_0 \longrightarrow C_0,$$

$$\begin{CD} C_1 \times_{C_0} C_1 @>{c}>> C_1 \\ @V{p_1}VV @VV{d_1}V \\ C_1 @>>{d_1}> C_0, \end{CD} \qquad \begin{CD} C_1 \times_{C_0} C_1 @>{c}>> C_1 \\ @V{p_2}VV @VV{d_0}V \\ C_1 @>>{d_0}> C_0, \end{CD}$$

$$\begin{CD} (C_1 \times_{C_0} C_1) \times_{C_0} C_1 @>{\sim}>> @. C_1 \times_{C_0} (C_1 \times_{C_0} C_1) \\ @V{c \times_{C_0} \mathrm{id}_{C_1}}VV @. @VV{\mathrm{id}_{C_1} \times_{C_0} c}V \\ C_1 \times_{C_0} C_1 @>>{c}> C_1 @<<{c}< C_1 \times_{C_0} C_1, \end{CD}$$

and

$$\begin{array}{ccccc} C_0 \times_{C_0} C_1 & \xrightarrow{s_0 \times_{C_0} \mathrm{id}_{C_1}} & C_1 \times_{C_0} C_1 & \xleftarrow{\mathrm{id}_{C_1} \times_{C_0} s_0} & C_1 \times_{C_0} C_0 \\ & {\scriptstyle p_2} \searrow & \downarrow {\scriptstyle c} & \swarrow {\scriptstyle p_1} & \\ & & A & & \end{array}$$

commute.

The objects C_0 and C_1 are called the (objects of) *objects* and *morphisms* of C respectively, while the morhisms s_0, d_0, d_1, and c are called the *identity-assigning*, *target*, *source*, and *composition* morphisms of C respectively. When convenient, we will denote C_0 and C_1 by $\operatorname{ob} C$ and $\operatorname{mor} C$ respectively. ⌟

Remark. Note that the pullbacks involved in particular have to exist.

Definition A.1.2. Let C and D be categories internal to a category $\mathscr{C}$. An *internal functor* $f\colon C \longrightarrow D$ is a pair of morphisms $f_0\colon C_0 \longrightarrow D_0$ and $f_1\colon C_1 \longrightarrow D_1$

making the squares containing consecutive columns and horizontal morphisms with the same label of

$$\begin{array}{ccccc}
C_0 & \underset{d_0}{\overset{d_1}{\underset{\xrightarrow{s_0}}{\leftleftarrows}}} & C_1 & \xleftarrow{c} & C_1 \times_{C_0} C_1 \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_1 \times_{D_0} f_1 \\
D_0 & \underset{d_2}{\overset{d_1}{\underset{\xrightarrow{s_0}}{\leftleftarrows}}} & D_1 & \xleftarrow[c]{} & D_1 \times_{D_0} D_1
\end{array}$$

commute. ⌟

Definition A.1.3. Let $\mathscr{C}$ be a category. The categories internal to $\mathscr{C}$ and internal functors between them form a category, which we will denote $\mathbf{Cat}(\mathscr{C})$. We call an object of $\mathbf{Cat}(\mathbf{Top})$ a *topological category.* ⌟

Remark. We identify $\mathbf{Cat}$ with $\mathbf{Cat}(\mathbf{Set})$.

Remark. If C is a category internal to a category $\mathscr{C}$ and $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is a functor that preserves pullbacks (e.g. a right adjoint), then F sends C to a category internal to $\mathscr{D}$. In general such a functor F induces a functor $\mathbf{Cat}(\mathscr{C}) \longrightarrow \mathbf{Cat}(\mathscr{D})$.

Definition A.1.4. Let C be a category internal to a category $\mathscr{C}$. The *internal nerve* $N^{\mathrm{int}}_\bullet(C)$ of C is the simplicial object of $\mathscr{C}$ (whose n^{th} object we denote $N^{\mathrm{int}}_n C$) defined by

$$N^{\mathrm{int}}_0 C = C_0, \quad N^{\mathrm{int}}_1 C = C_1, \quad \text{and} \quad N^{\mathrm{int}}_n C = C_1^{\times_{C_0} n}$$

for all $n \geq 2$, with the evident face- and degeneracy-maps coming from the structure morphisms of C. The internal nerve forms a functor $\mathbf{Cat}(\mathscr{C}) \longrightarrow \mathrm{s}\mathscr{C}$ ⌟

A.2. Topological categories and their classifying spaces

Definition A.2.1. Let $X_\bullet \in \mathbf{sTop}$. We define the *realization* and *fat realization* of $X_\bullet$ as the coends

$$|X_\bullet| := \int^{[n]\in\mathbf{\Delta}} X_n \times \Delta^n_{\mathbf{Top}} \quad \text{and} \quad \|X_\bullet\| := \int^{[n]\in\mathbf{\Delta}_+} X_n \times \Delta^n_{\mathbf{Top}}$$

respectively, where $\Delta^\bullet_{\mathbf{Top}}$ denotes the usual cosimplicial object in $\mathbf{Top}$ and $\mathbf{\Delta}_+$ the wide subcategory of $\mathbf{\Delta}$ spanned by injective functions. For $X_\bullet \in \mathbf{sSet}$ we let $|X_\bullet| := \left|X^\delta_\bullet\right|$ and $\|X_\bullet\| := \left\|X^\delta_\bullet\right\|$, where $(-)^\delta\colon \mathbf{sSet} \longrightarrow \mathbf{sTop}$ is the functor induced by the functor $\mathbf{Set} \longrightarrow \mathbf{Top}$ giving the discrete topology (the left adjoint of the forgetful functor). ⌟

Definition A.2.2. For a topological category C, we define BC by

$$BC := \left\|N^{\mathrm{int}}_\bullet C\right\|$$

and call it the (*fat*) *classifying space* of C. ⌟

Definition A.2.3. If a continuous functor $F\colon C \longrightarrow D$ in $\mathbf{Cat}(\mathbf{Top})$ satisfies that

$$N^{\mathrm{int}}_k F\colon N^{\mathrm{int}}_k C \longrightarrow N^{\mathrm{int}}_k D$$

is a homotopy equivalence for all $k \in \mathbb{N}_0$, we call F a *level-wise homotopy equivalence* (*of categories*). ⌟

Lemma A.2.4. *Let $f_\bullet\colon A_\bullet \longrightarrow B_\bullet$ be a map of simplicial spaces. If $f_n\colon A_n \longrightarrow B_n$ is a homotopy equivalence for all $n \in \mathbb{N}_0$, then the induced map*

$$\|f_\bullet\|\colon \|A_\bullet\| \longrightarrow \|B_\bullet\|$$

is a homotopy equivalence.

Proof. This is [10, Proposition A.1 (ii)]. □

Corollary A.2.5. *If $F\colon C \longrightarrow D$ is a level-wise homotopy equivalence of categories, then the induced map $BF\colon BC \longrightarrow BD$ is a homotopy equivalence.*

A.3. Symmetric monoidal categories

Definition A.3.1. A *monoidal category* $(\mathscr{C}, \otimes, \mathbb{1})$ consists of the following data: a category $\mathscr{C}$, an object $\mathbb{1} \in \mathscr{C}$, and a functor $\otimes\colon \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ along with isomorphisms

$$a\colon (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z), \quad \lambda\colon \mathbb{1} \otimes x \longrightarrow x, \quad \text{and} \quad \rho\colon x \otimes \mathbb{1} \longrightarrow x$$

natural in $x, y, z \in \mathscr{C}$, making certain diagrams found in [5, chapter VII] commute. If, furthermore, a, λ and ρ are the identity morphisms, we say that the monoidal category is *strict*. ⌟

Definition A.3.2. Let $\mathscr{C}$ be a monoidal category. A *braiding* on $\mathscr{C}$ is an isomorphism $b\colon x \otimes y \longrightarrow y \otimes x$, natural in $x, y \in \mathscr{C}$, making certain diagrams found in [5, pp. 252–253] commute. ⌟

Definition A.3.3. A *symmetric monoidal category* $(\mathscr{C}, \otimes, \mathbb{1})$ is a monoidal category $(\mathscr{C}, \otimes, \mathbb{1})$ with a braiding b, such that the composition

$$x \otimes y \xrightarrow{b_{x,y}} y \otimes x \xrightarrow{b_{y,x}} x \otimes y$$

is the identity for all $x, y \in \mathscr{C}$. ⌟

Definition A.3.4. A cartesian (monoidal) category is a symmetric monoidal category whose monoidal structure is given by the category theoretic product in the category. ⌟

Definition A.3.5. Let $\mathscr{C}$ be a monoidal category. A *monoid* (object) (M, μ, e) of $\mathscr{C}$ is an object $M \in \mathscr{C}$ together with morphisms

$$\mu\colon M \otimes M \longrightarrow M, \quad \text{and} \quad e\colon \mathbb{1} \longrightarrow M$$

such that the diagrams

(*associativity*)
$$\begin{array}{ccccc} (M \otimes M) \otimes M & & \xrightarrow{\quad a \quad} & & M \otimes (M \otimes M) \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_M} & & & & \downarrow{\scriptstyle \mathrm{id}_M \otimes \mu} \\ M \otimes M & \xrightarrow[\mu]{} & M & \xleftarrow[\mu]{} & M \otimes M \end{array}$$

and

(*left/right unitality*)
$$\begin{array}{ccccc} \mathbb{1} \otimes M & \xrightarrow{e \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes e} & M \otimes \mathbb{1} \\ & {\scriptstyle \lambda} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \rho} & \\ & & M & & \end{array}$$

commute. ⌟

Remark. In case $\mathscr{C}$ is cartesian, we can define the notion of a *group* object of $\mathscr{C}$.

Definition A.3.6. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. An *internal hom-functor* to $\mathscr{C}$ is a functor $\underline{\mathrm{Hom}}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{C}$ satisfying that for each $x \in \mathscr{C}$, the functors

$$(-) \otimes x\colon \mathscr{C} \longrightarrow \mathscr{C} \quad \text{and} \quad \underline{\mathrm{Hom}}(x, -)\colon \mathscr{C} \longrightarrow \mathscr{C}$$

form an adjoint pair $(-) \otimes x \dashv \underline{\mathrm{Hom}}(x, -)$. ⌟

A.4. Actions

In this section we define actions of a *monoid*. The reader not familiar with monoids, can think of the special case of groups instead.

We first give an "external" definition of an action of a group.

Definition A.4.1 (External monoid action)**.** Let M be a monoid, $\mathscr{C}$ a category, and $c \in \mathscr{C}$. A *left* (resp. *right*) *action* of M on c is a covariant (resp. contravariant) functor

$$\alpha\colon \mathcal{B}M \longrightarrow \mathscr{C} \qquad (\text{resp. } \alpha\colon \mathcal{B}M^{\mathrm{op}} \longrightarrow \mathscr{C})$$

sending the object of $\mathcal{B}M$ to c. We denote $\alpha(m)$ by $m.(-)$ (resp. $(-).m$) for all $m \in M$. ⌟

Remark. In the literature, there is a notion of an action being *effective*. With our external definition, this is simply the property, that the action is *faithful*.

Definition A.4.2 (Internal monoid action)**.** Let $\mathscr{C}$ be a category, M a monoid object in $\mathscr{C}$, and $X \in \mathscr{C}$. A *left action* of M on X is a morphism

$$\alpha\colon M \otimes X \longrightarrow X$$

such that the diagram

$$\begin{array}{ccccccc}
(M \otimes M) \otimes X & \xrightarrow{a} & M \otimes (M \otimes X) & \xrightarrow{\mathrm{id}_M \otimes \alpha} & M \otimes X & \xleftarrow{e \otimes \mathrm{id}_X} & \mathbb{1} \otimes X \\
\downarrow{\scriptstyle \mu \otimes \mathrm{id}_X} & & & & \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \lambda} & \\
M \otimes X & & \xrightarrow{\qquad\qquad \alpha \qquad\qquad} & & X & &
\end{array}$$

commutes. We let the reader define the evident notion of a *right action*. ⌟

Remark. Note that a group object in a cartesian category contains a monoid object. We define the action of a group object, to be an action of the monoid it contains (of the same "dexterity/handedness").

A.5. Operads

These were first defined in [8] though only in **Top**, but the definition is an obvious generalization. Our definitions in this section come from [11, §1.1.1–1.1.3],

Notation. We let Σ_n denote the symmetric group on the set of n elements.

Definition A.5.1. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. An *operad* $\mathcal{O}$ in $\mathscr{C}$ is a collection of objects $\mathcal{O}(n)$ indexed by $n \in \mathbb{N}_0$ equipped with (external) right actions of Σ_n on $\mathcal{O}(n)$ and a morphism $\eta\colon \mathbb{1} \longrightarrow \mathcal{O}(1)$ along with product morphisms

$$\gamma\colon \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \longrightarrow \mathcal{O}(n_1 + \cdots + n_k)$$

which are associative, unital and Σ-equivariant in the sense that the diagrams
(*associativity*)

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes \bigotimes_{i=1}^{k} \mathcal{O}(n_i) \otimes \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{O}(m_{i,j}) & \xrightarrow{\gamma \otimes \mathrm{id}} & \mathcal{O}\big(\sum_{i=1}^{k} n_i\big) \otimes \bigotimes_{i=1}^{k} \bigotimes_{j=1}^{n_i} \mathcal{O}(m_{i,j}) \\
\downarrow{\scriptstyle \mathrm{shuffle}} & & \\
\mathcal{O}(k) \otimes \bigotimes_{i=1}^{k} \Big(\mathcal{O}(n_i) \otimes \bigotimes_{j=1}^{n_i} \mathcal{O}(m_{i,j}) \Big) & & \downarrow{\scriptstyle \gamma} \\
\downarrow{\scriptstyle \mathrm{id} \otimes \bigotimes_{i=1}^{k} \gamma} & & \\
\mathcal{O}(k) \otimes \bigotimes_{i=1}^{k} \mathcal{O}\big(\sum_{j=1}^{n_i} m_{i,j}\big) & \xrightarrow{\gamma} & \mathcal{O}\big(\sum_{i=1}^{k} \sum_{j=1}^{n_i} m_{i,j}\big),
\end{array}$$

(Σ-*equivariance*)

$$\begin{array}{ccc} \mathcal{O}(n)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k) & \xrightarrow{.\sigma\otimes\sigma^{-1}} & \mathcal{O}(n)\otimes\mathcal{O}(n_{\sigma(1)})\otimes\cdots\otimes\mathcal{O}(n_{\sigma(k)}) \\ \downarrow\gamma & & \downarrow\gamma \\ \mathcal{O}(n_1+n_2+\cdots+n_k) & \xrightarrow{.\sigma(n_1,n_2,\ldots,n_k)} & \mathcal{O}(n_1+n_2+\cdots+n_k), \end{array}$$

$$\begin{array}{ccc} \mathcal{O}(n)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k) & \xrightarrow{\mathrm{id}\otimes.\tau_1\otimes\cdots\otimes.\tau_k} & \mathcal{O}(n)\otimes\mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k) \\ \downarrow\gamma & & \downarrow\gamma \\ \mathcal{O}(n_1+n_2+\cdots+n_k) & \xrightarrow{.(\tau_1\oplus\cdots\oplus\tau_k)} & \mathcal{O}(n_1+n_2+\cdots+n_k), \end{array}$$

(*unitality*)
$$\begin{array}{ccc} \mathbb{1}\otimes\mathcal{O}(n) & \xrightarrow{\lambda} & \mathcal{O}(n) \\ {\scriptstyle\eta\otimes\mathrm{id}}\downarrow & \nearrow_{\gamma} & \\ \mathcal{O}(1)\otimes\mathcal{O}(n), & & \end{array} \quad\text{and}\quad \begin{array}{ccc} \mathcal{O}(n)\otimes\mathbb{1}^{\otimes n} & \xrightarrow{\rho^n} & \mathcal{O}(n) \\ {\scriptstyle\mathrm{id}\otimes\eta^{\otimes n}}\downarrow & \nearrow_{\gamma} & \\ \mathcal{O}(n)\otimes\mathcal{O}(1)^{\otimes n} & & \end{array}$$

commute. ⌟

Definition A.5.2. Let $(\mathscr{C},\otimes,\mathbb{1})$ be a symmetric monoidal category with internal hom-functor. The *endomorphism operad* $\mathcal{E}_x$ of an object $x\in\mathscr{C}$ is the operad with $\mathcal{E}_x(k)=\underline{\mathrm{Hom}}(x^{\otimes k},x)$ and composition γ given by the composition in $\mathscr{C}$. If $\mathcal{O}$ is an operad in $\mathscr{C}$, then an *algebra* over $\mathcal{O}$ is an object $x\in\mathscr{C}$ and a morphism of operads[4] $\mathcal{O}\longrightarrow\mathcal{E}_x$. ⌟

Definition A.5.3. Let $\mathring{D}^n$ denote the n-dimensional open unit-disc of $\mathbb{R}^n$. *The little n-discs operad* $\mathcal{D}_n$ is the operad in **Top** with $\mathcal{D}_n(k)$ the space of embeddings

$$f\colon \coprod_{i=1}^{k}\mathring{D}^n\longrightarrow\mathring{D}^n,$$

which restrict to a dilation followed by a translation on each component, and with composition map γ given by composition:

$$\gamma(f,g_1,\ldots,g_k)=\coprod_{i=1}^{k}\coprod_{j=1}^{n_k}\mathring{D}^n\xrightarrow{\coprod_{i=1}^k g_i}\coprod_{i=1}^{k}\mathring{D}^n\xrightarrow{f}\mathring{D}^n.$$ ⌟

Remark. There is an obvious analogue of the little n-discs operad, called the little n-cubes operad, using open n-cubes (i.e. $(0,1)^n$) instead of n-discs, which is, of course, isomorphic to $\mathcal{D}_n$.

Definition A.5.4. An $\mathbb{E}_n$-algebra, is an algebra over $\mathcal{D}_n$. ⌟

[4]which is what you think it is.

Appendix B. Sheaves

This appendix gives the classical definition of sheaves with values in a complete category on a topological space, and their stalks.

B.1. Classic sheaves

Definition B.1.1. For a topological space X we denote by X_{Zar} the poset category of the topology of X (i.e. the open sets partially ordered by inclusion). ⌟

Definition B.1.2. Let X be a topological space. A *$\mathscr{C}$-valued presheaf* on X is a $\mathscr{C}$-valued presheaf (in the categorical sense) on X_{Zar}. We write $\mathrm{PSh}(X;\mathscr{C}) := \mathrm{PSh}(X_{\mathrm{Zar}};\mathscr{C})$. For a presheaf $\mathscr{F} \in \mathrm{PSh}(X;\mathscr{C})$ we write $(-)|_B^A \colon \mathscr{F}(A) \longrightarrow \mathscr{F}(B)$ for $\mathscr{F}(B \subset A)$ (suppressing $\mathscr{F}$ – we will also often suppress A, when it is clear from context). ⌟

Definition B.1.3. Let $\mathscr{F}$ be a $\mathscr{C}$-valued presheaf on X and $A \subset X$. The *stalk of $\mathscr{F}$ at A* is the filtered colimit

$$\mathscr{F}_A := \operatorname*{colim}_{U \in X_{\mathrm{Zar}}^{\mathrm{op}} \colon A \subset U} \mathscr{F}(U)$$

if it exists. An element of $\mathscr{F}_A$ is called a *germ of $\mathscr{F}$ near A*. In case $A = \{x\}$ is a singleton we write $\mathscr{F}_x := \mathscr{F}_{\{x\}}$ and call the elements of $\mathscr{F}_x$ *germs of $\mathscr{F}$ at x* instead. For any $U \in X_{\mathrm{Zar}}$ with $A \subset U$ we thus have a canonical map $\mathscr{F}(U) \longrightarrow \mathscr{F}_A$. ⌟

Remark. Note that, writing this out in the case $\mathscr{C} = \mathbf{Set}$, we can model $\mathscr{F}_A$ by

$$\mathscr{F}_A = \{(U, f) \mid U \in X_{\mathrm{Zar}}, A \subset U, f \in \mathscr{F}(U)\}/\sim,$$

where $\sim$ is the equivalence relation defined by $(U, f) \sim (U', f')$ if there exists $V \in X_{\mathrm{Zar}}$ with $A \subset V$ such that $V \subset U \cap U'$ and $f|_V = f'|_V$. For $f \in \mathscr{F}(U)$ with $A \subset U$ we write f_A (resp. f_x) for the image of f under the canonical morphism $\mathscr{F}(U) \longrightarrow \mathscr{F}_A$ (given by $f_A = (U, f)_{/\sim}$).

Definition B.1.4. Let X be a topological space. A *sheaf of sets*[5] on X is a contravariant functor $\mathscr{F} \colon X_{\mathrm{Zar}}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, which satisfies the following *sheaf condition*: If $f_i \in \mathscr{F}(U_i)$ is a collection of elements for an open cover $\{U_i\}_{i \in I}$ of an open subset $U \subset X$, which satisfies $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a unique $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$. ⌟

Classic examples are smooth and continuous maps to a fixed space.

B.2. Sheaves with values in a complete category

We note that we can write the sheaf condition as that the diagram

$$\mathscr{F}(U) \xrightarrow{\ e\ } \prod_{i \in I} \mathscr{F}(U_i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j)$$

is an equalizer, where a, b are the unique maps such that

$$\mathrm{pr}_{(i,j)} \circ a = (\mathrm{pr}_i(-))|_{U_i \cap U_j}^{U_i}$$
$$\mathrm{pr}_{(i,j)} \circ b = (\mathrm{pr}_j(-))|_{U_i \cap U_j}^{U_j}$$

for all $(i, j) \in I \times I$ and e is $((-)|_{U_i}^{U})_{i \in I}$.

[5] or **Set**-valued sheaf on X

Definition B.2.1. Let X be a topological space and $\mathscr{C}$ a complete category. A $\mathscr{C}$-valued sheaf on X is a contravariant functor $\mathscr{F}\colon X_{\mathrm{Zar}}^{\mathrm{op}} \longrightarrow \mathscr{C}$ such that the diagram

$$\mathscr{F}(U) \xrightarrow{\ e\ } \prod_{i\in I} \mathscr{F}(U_i) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathscr{F}(U_i \cap U_j) \tag{B.1}$$

is an equalizer for any open covering $\{U_i\}_{i\in I}$ of an open subset $U \subset X$, where a, b are the unique maps such that

$$\mathrm{pr}_{(i,j)} \circ a = \mathrm{res}^{U_i}_{U_i\cap U_j} \circ \mathrm{pr}_i$$
$$\mathrm{pr}_{(i,j)} \circ b = \mathrm{res}^{U_j}_{U_i\cap U_j} \circ \mathrm{pr}_j$$

for all $(i, j) \in I \times I$. The category of $\mathscr{C}$-valued sheaves on X is the full subcategory $\mathrm{Sh}(X; \mathscr{C}) \hookrightarrow \mathrm{PSh}(X; \mathscr{C})$ spanned by the $\mathscr{C}$-valued sheaves. ⌟

Appendix C. The sheaf of Galatius–Randal-Williams

Here we will define the **Top**-valued sheaves Ψ_d on $\mathbb{R}^n$ of d-dimensional manifolds by Galatius–Randal-Williams following [4, section 2] in an almost one-to-one fashion other than that we skip tangential structures. This means any errors introduced are our own. In contrast to [4], our exposition is more verbose. To do this, we first define its underlying sheaf of sets denoting it by Ψ^{s}_d.

Definition C.0.1. Let $d, n \in \mathbb{N}_0$. For $U \subset \mathbb{R}^n$ open, let $\Psi^{\mathrm{s}}_{d,n}(U)$ be the set of all d-dimensional, smooth submanifolds $M \subset U$ without boundary such that M is a closed subset of U. For open subspaces $V \subset U$ of $\mathbb{R}^n$, define the restriction function $\Psi^{\mathrm{s}}_{d,n}(U) \longrightarrow \Psi^{\mathrm{s}}_{d,n}(V)$ by $M \mapsto M \cap V$ for all $M \in \Psi^{\mathrm{s}}_{d,n}(U)$. The restriction functions assemble to make $\Psi^{\mathrm{s}}_{d,n}$ a sheaf of sets on $\mathbb{R}^n$. We will suppress d and n from the notation whenever convenient. ⌟

C.1. Construction of the topology

In this section, we will define a topology on $\Psi^{\mathrm{s}}_{d,n}(U)$ for all $d, n \in \mathbb{N}_0$ and all open subspaces $U \subset \mathbb{R}^n$ giving a **Top**-valued sheaf which we will denote $\Psi_{d,n}$. Since we do this for all $d, n \in \mathbb{N}_0$, we will consider d, n, and $U \subset \mathbb{R}^n$ fixed and suppress d and n from notation. We will define this topology in terms of two preliminary topologies on $\Psi^{\mathrm{s}}(U)$, namely the compactly supported topology and the K-topologies.

C.1.1. *The compactly supported topology.* First, we define the topological spaces $C^\infty_{\mathrm{c}}(M, \mathbb{R})$ and $\Gamma_{\mathrm{c}}(\nu_M)$ for all $M \in \Psi^{\mathrm{s}}(U)$.

Definition C.1.1. Let $M \in \Psi^{\mathrm{s}}(U)$ and let $C^\infty_{\mathrm{c}}(M, \mathbb{R})$ denote the set of compactly supported, real-valued, smooth functions on M. For a smooth map $\epsilon\colon M \longrightarrow (0, \infty)$, $r \in \mathbb{N}_0$, and an r-tuple $X = (X_1, \dots, X_r)$ of smooth vector fields[6] on M, let $B(\epsilon, X)$ denote the set

$$B(\epsilon, X) := \{f \in C^\infty_{\mathrm{c}}(M, \mathbb{R}) \mid |(X_1 \cdots X_r f)(x)| < \epsilon(x) \text{ for all } x \in M\}.$$

We give $C^\infty_{\mathrm{c}}(M, \mathbb{R})$ the unique topology for which the collection

$$\left\{ f + B(\epsilon, X) \;\middle|\; \begin{array}{l} f \in C^\infty_{\mathrm{c}}(M, \mathbb{R}), \\ \epsilon\colon M \longrightarrow (0, \infty) \text{ is a smooth map, and} \\ X \text{ is a finite tuple of smooth vector fields on } M \end{array} \right\}$$

of subsets form a sub-basis. ⌟

Definition C.1.2. For $M \in \Psi^{\mathrm{s}}(U)$, we let ν_M denote the normal bundle of M, which we consider to be a subbundle of the trivial n-dimensional $\mathbb{R}$-vectorbundle ε^n over M, namely the orthogonal complement of the tangent bundle. We let $\Gamma_{\mathrm{c}}(\nu_M)$ denote the set of compactly supported, smooth sections of ν_M. Post-composition with the i^{th} coordinate-projection on the fibres of ε^n yields a linear map

$$\begin{gathered} \Gamma_{\mathrm{c}}(\nu_M) \longrightarrow C^\infty_{\mathrm{c}}(M, \mathbb{R}), \quad \text{defined by} \\ (s\colon M \longrightarrow \mathbb{R}^n) \longmapsto (s \circ \mathrm{pr}_i\colon M \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}). \end{gathered}$$

[6] Note that r is allowed to be 0 and hence X is allowed to be empty.

These n coordinate maps assemble to a linear injection $\Gamma_{\mathrm{c}}(\nu_M) \hookrightarrow C^\infty_{\mathrm{c}}(M,\mathbb{R})^{\oplus n}$. Give $C^\infty_{\mathrm{c}}(M,\mathbb{R})^{\oplus n}$ the product topology. We give $\Gamma_{\mathrm{c}}(\nu_M)$ the initial topology w.r.t. this injection, i.e. the unique topology making this injection an embedding. ⌟

We now define the compactly supported topology on $\Psi^{\mathrm{s}}(U)$.

Definition C.1.3. The proof of the tubular neighbourhood theorem[7] provides for every $M \in \Psi^{\mathrm{cs}}(U)$ a partially defined function

$$c_M \colon \Gamma_{\mathrm{c}}(\nu_M) \dashrightarrow \Psi^{\mathrm{s}}(U)$$

defined on an open neighbourhood of M and given by $s \mapsto \{x + s(x) \mid x \in M\}$, where defined. The *compactly supported topology* on $\Psi^{\mathrm{s}}(U)$ is the unique topology on $\Psi^{\mathrm{s}}(U)$ making the functions

$$c_M \colon \Gamma_{\mathrm{c}}(\nu_M) \dashrightarrow \Psi^{\mathrm{s}}(U)$$

open embeddings for every $M \in \Psi^{\mathrm{s}}(U)$. We let $\Psi^{\mathrm{cs}}(U)$ denote $\Psi^{\mathrm{s}}(U)$ equipped with the compactly supported topology. ⌟

Warning. It will be important for the topology and well-definedness of maps, that we are only considering *compactly supported* maps. This gives us another warning: The compactly supported topology Ψ^{cs} does not form a presheaf as the restriction functions are not continuous.

C.1.2. *The K-topology.* For each compact subset $K \subset U$ we will define the so-called "K-topology" on $\Psi^{\mathrm{s}}(U)$. In order to do this we define a preliminary topology on the stalk Ψ^{s}_K, as defined in definition B.1.3, of Ψ^{s} near K, which depends on U.

Definition C.1.4. Let $K \subset U$ be compact. We give Ψ^{s}_K the quotient topology of $\Psi^{\mathrm{cs}}(U)$ with respect to the canonical (surjective) projection to the stalk.

$$\Psi^{\mathrm{cs}}(U) \xrightarrow{\pi_K} \Psi^{\mathrm{s}}_K.$$

This topology depends on U. Therefore, we will denote Ψ^{s}_K equipped with this topology by $(\Psi^{\mathrm{s}}|_U)_K$ (i.e. the stalk of the restriction of Ψ^{s} to U near K). ⌟

Definition C.1.5. The *K-topology* on $\Psi^{\mathrm{s}}(U)$ is the initial topology with respect to the canonical projection function

$$\pi_K \colon \Psi^{\mathrm{s}}(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_K.$$

We let $\Psi^K(U)$ denote $\Psi^{\mathrm{s}}(U)$ equipped with the K-topology. ⌟

We use the following lemma in the definition of the myopic topology in the following section.

Lemma C.1.6. *If $K \subset L$ are compact subsets of U, then the identity function*

$$\Psi^L(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U)$$

is continuous (i.e. the L-topology is finer than the K-topology).

Proof. Using functoriality of taking stalks, we get the commutative diagram

$$\begin{array}{ccccc}
\Psi^L(U) & \xrightarrow{\pi_L} & (\Psi^{\mathrm{s}}|_U)_L & \xleftarrow{\pi_L} & \Psi^{\mathrm{cs}}(U) \\
{\scriptstyle \mathrm{id}^{\mathrm{s}}}\downarrow & \searrow^{\pi_K} & \downarrow{\scriptstyle (-)_{K\subset L}} & \swarrow_{\pi_K} & \\
\Psi^K(U) & \xrightarrow[\pi_K]{} & (\Psi^{\mathrm{s}}|_U)_K & &
\end{array}$$

in **Set**. All the projection functions except $\pi_K \colon \Psi^L(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_K$ are continuous by definition. Since $\pi_L \colon \Psi^{\mathrm{cs}}(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_L$ is a quotient map, the function labelled $(-)_{K\subset L}$ is continuous and so $\pi_K \colon \Psi^L(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_K$ is continuous by commutativity. It now follows that id^{s} is continuous by definition of $\Psi^K(U)$. □

[7] [1], Theorem 11.4 and the paragraph after its proof, since M may not be compact.

C.1.3. *The myopic topology.* Finally, we will define what the author has taken to calling the "myopic" (i.e. nearsighted) topology on $\Psi^{\mathrm{s}}(U)$.

Definition C.1.7. The *myopic* topology on $\Psi^{\mathrm{s}}(U)$ is the initial topology with respect to the collection

$$\{\Psi^{\mathrm{s}}(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U)\}_{K \underset{\mathrm{cpt.}}{\subset} U}$$

of functions, and we let $\Psi(U)$ denote $\Psi^{\mathrm{s}}(U)$ equipped with this topology. ⌟

Remark. This means we can write

$$\Psi(U) := \lim_{K \in (U_{\mathrm{cpt}})^{\mathrm{op}}} \Psi^K(U),$$

where U_{cpt} is the poset category of compact subsets of U ordered by inclusion and where all maps in the limit are the identity function.

C.2. As a sheaf of topological spaces

Having defined Ψ in the previous section, we will in this section show that it forms a **Top**-valued sheaf.

Lemma C.2.1. *Let $V \subset U$ be open subsets of $\mathbb{R}^n$. Then the restriction function $r\colon \Psi^{\mathrm{cs}}(U) \longrightarrow \Psi^{\mathrm{cs}}(V)$ is open.*

Proof. For each $M \in \Psi^{\mathrm{cs}}(U)$ we have the diagram

$$\begin{array}{ccc} \Gamma_{\mathrm{c}}(\nu_M) & \overset{c_M}{\dashrightarrow} & \Psi^{\mathrm{cs}}(U) \\ {\scriptstyle z}\uparrow & & \downarrow{\scriptstyle r} \\ \Gamma_{\mathrm{c}}(\nu_{M\cap V}) & \underset{c_{M\cap V}}{\dashrightarrow} & \Psi^{\mathrm{cs}}(V), \end{array} \tag{C.1}$$

where z denotes the map extending sections with 0. It should be clear that z is both well-defined (remember the sections are compactly supported) and continuous. Let $A \subset \Psi^{\mathrm{cs}}(U)$ be open. For each $M \in A$ define

$$A_M := c_{M\cap V}(z^{-1}(c_M^{-1}(A))) \subset \Psi^{\mathrm{cs}}(V)$$

and note that A_M is open, since z and c_M are continuous and $c_{M\cap V}$ is open. Note that $r(M) \in A_M$ for each $M \in A$, since $c_M(z(0)) = c_M(0) = M$ and $c_{M\cap V}(0) = M \cap V = r(M)$. Further noting, that $r|_{\mathrm{im}\, z_M} \circ z_M = \mathrm{id}_{A_M}$ gives $A_M \subset r(A)$ for all $M \in A$ allowing us to conclude that

$$r(A) = \bigcup_{M \in A} A_M,$$

which implies r is open. □

Lemma C.2.2. *Let $K \subset U$ be compact, $\epsilon \in \mathbb{R}$ such that $0 < 4\epsilon \le \mathrm{dist}(K, \mathbb{R}^n \backslash U)$, $\lambda\colon U \longrightarrow [0,1]$ a smooth function satisfying*

$$\lambda(x) = \begin{cases} 1 & \mathrm{dist}(x,K) \le 2\epsilon \\ 0 & \mathrm{dist}(x,K) \ge 3\epsilon, \end{cases}$$

$V \subset U$ any open subset containing the support of λ, and $M \in \Psi^{\mathrm{cs}}(U)$. The function $\bar\lambda\colon \Gamma_{\mathrm{c}}(\nu_M) \longrightarrow \Gamma_{\mathrm{c}}(\nu_{M\cap V})$ given by multiplication by λ is continuous. If we let $z\colon \Gamma_{\mathrm{c}}(\nu_{M\cap V}) \longrightarrow \Gamma_{\mathrm{c}}(\nu_M)$ denote the continuous function given by extending sections with 0, then the diagram

$$\begin{array}{ccccc} \Gamma_{\mathrm{c}}(\nu_{M\cap V}) & \xrightarrow{z} & \Gamma_{\mathrm{c}}(\nu_M) & \overset{c_M}{\dashrightarrow} & \Psi^{\mathrm{cs}}(U) \\ {\scriptstyle \bar\lambda}\uparrow & & & & \downarrow{\scriptstyle \pi_K} \\ \Gamma_{\mathrm{c}}(\nu_M) & \overset{c_M}{\dashrightarrow} & \Psi^{\mathrm{cs}}(U) & \xrightarrow{\pi_K} & (\Psi^{\mathrm{s}}|_U)_K \end{array}$$

commutes after restricting to an open neighbourhood of the 0*-section.*

Proof. It should be clear, that $\bar{\lambda}$ is continuous.

Let $0 \in \Gamma_c(\nu_M)$ denote the 0-section of ν_M. Let U_1 be an open neighbourhood of 0 on which c_M is defined. Note that $\bar{\lambda}(z(0)) = 0$, so 0 is an element of $U_1 \cap (\bar{\lambda} \circ z)^{-1}(U_1)$. Now define

$$U_0 \coloneqq U_1 \cap (\bar{\lambda} \circ z)^{-1}(U_1) \cap \{s \in \Gamma_c(\nu_M) \mid |s(x)| < \epsilon \text{ for all } x \in M\}$$

and note that U_0 is an open neighbourhood of 0. We will now show that the diagram commutes when restricted to U_0.

For each $r \in \mathbb{R}_{>0}$ let K_r denote the open set $\{x \in U \mid \operatorname{dist}(x, K) < r\}$. Let $s \in U_0$. We have to show that $c_M(s)$ and $c_M(z(\bar{\lambda}(s)))$ agree on an open neighbourhood of K. We show, they agree on K_ϵ. Assume, for the sake of contradiction, that $c_M(s) \cap K_\epsilon \neq c_M(z(\bar{\lambda}(s))) \cap K_\epsilon$. This allows us to assume without loss of generality, that there exists $x \in c_M(s) \cap K_\epsilon$ such that $x \notin c_M(z(\bar{\lambda}(s))) \cap K_\epsilon$ and fix such an x. The point x is of the form $x = x' + s(x')$ for a unique point $x' \in M$, since $x \in c_M(s)$. We cannot have $x' \in K_{2\epsilon}$ because $s|_{M \cap K_{2\epsilon}} = z(\bar{\lambda}(s))|_{M \cap K_{2\epsilon}}$. This means $\operatorname{dist}(x', K) \geq 2\epsilon$. But, because $x \in K_\epsilon$, we must have $\epsilon \leq |x - x'| = |s(x')|$, so since $s \in U_0$, which yields $|s(x')| < \epsilon$, we have a contradiction. □

Lemma C.2.3. *The quotient map* $\pi_K \colon \Psi^{\mathrm{cs}}(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_K$ *is open.*

Proof. We omit the proof due to lack of time. This is [4, Lemma 2.5]. □

Lemma C.2.4. *Let* $K \subset U$ *be compact. If* $V \subset U$ *is an open subset containing* K*, then the function* $\rho \colon (\Psi^{\mathrm{s}}|_U)_K \longrightarrow (\Psi^{\mathrm{s}}|_V)_K$ *induced on stalks by the restriction* $r \colon \Psi^{\mathrm{cs}}(U) \longrightarrow \Psi^{\mathrm{cs}}(V)$ *is an open embedding.*

Proof. It suffices to show, that ρ is injective, continuous, and open. That ρ is injective is clear.

We will check continuity of ρ on an open cover. By definition $\{\operatorname{im} c_M\}_{M \in \Psi^{\mathrm{cs}}(U)}$ is an open cover of $\Psi^{\mathrm{cs}}(U)$, so since π_K is open and surjective, $\{\operatorname{im} \pi_K \circ c_M\}_{M \in \Psi^{\mathrm{cs}}(U)}$ is an open cover of $(\Psi^{\mathrm{s}}|_U)_K$. This is the open cover, we will check continuity of ρ on. Let $M \in \Psi^{\mathrm{cs}}(U)$ and pick $\bar{\lambda} \colon \Gamma_c(\nu_M) \longrightarrow \Gamma_c(\nu_{M \cap V})$ as in lemma C.2.2. We get the diagram

$$\begin{array}{ccccc} \Gamma_c(\nu_M) & \overset{c_M}{\dashrightarrow} & \Psi^{\mathrm{cs}}(U) & \overset{\pi_K}{\longrightarrow} & (\Psi^{\mathrm{s}}|_U)_K \\ \downarrow \bar{\lambda} & & r \downarrow & & \downarrow \rho \\ \Gamma_c(\nu_{M \cap V}) & \overset{c_{M \cap V}}{\dashrightarrow} & \Psi^{\mathrm{cs}}(V) & \overset{\pi_K}{\longrightarrow} & (\Psi^{\mathrm{s}}|_V)_K \end{array}$$

in **Set**, where the right-hand square commutes (by definition) and, for a sufficiently small neighbourhood of the 0-section of ν_M, the outer rectangle commutes by lemma C.2.2 and commutativity of (C.1). We know, all labeled functions except r and ρ are continuous, and so $\rho \circ \pi_K \circ c_M$ is continuous. Thus, since the (co)restriction of $\pi_K \circ c_M$ is a quotient map,[8] we get that ρ is continuous on $\operatorname{im} \pi_K \circ c_M$.

That ρ is open follows from commutativity of the right-hand square; ρ precomposed with the surjective map π_K is open (since both r and π_K are open), and so ρ is open. □

Lemma C.2.5. *The construction* Ψ *forms a presheaf.*

[8]this is because π_K is a quotient map and c_M is a quotient map after corestricting to its image and quotient maps are closed under composition and closed under restricting to a subspace while corestricting to the image of that subspace.

Proof. Let $V \subset U$ be open subsets of $\mathbb{R}^n$. We will show that the restriction $(-)|_V : \Psi(U) \longrightarrow \Psi(V)$ is continuous. By definition, $(-)|_V : \Psi(U) \longrightarrow \Psi(V)$ is continuous if and only if

$$\Psi(U) \xrightarrow{(-)|_V} \Psi(V) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(V)$$

is continuous for all compact subsets $K \subset V$. Let K be a compact subset of V. Since

$$\begin{array}{ccc} \Psi(U) & \xrightarrow{(-)|_V} & \Psi(V) \\ {\scriptstyle \mathrm{id}^{\mathrm{s}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}^{\mathrm{s}}} \\ \Psi^K(U) & \xrightarrow[(-)|_V]{} & \Psi^K(V) \end{array}$$

is commutative and $\mathrm{id}^{\mathrm{s}} : \Psi(U) \longrightarrow \Psi^K(U)$ is continuous (by definition), it suffices to show that $(-)|_V : \Psi^K(U) \longrightarrow \Psi^K(V)$ is continuous. Continuity of

$$(-)|_V : \Psi^K(U) \longrightarrow \Psi^K(V)$$

follows from commutativity of

$$\begin{array}{ccc} \Psi^K(U) & \xrightarrow{(-)|_V} & \Psi^K(V) \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi_K} \\ (\Psi^{\mathrm{s}}|_U)_K & \xrightarrow[\rho]{} & (\Psi^{\mathrm{s}}|_V)_K \end{array}$$

and that $\Psi^K(V)$ is initial w.r.t. π_K. □

Lemma C.2.6. *Let $I \neq \emptyset$ be a finite set. If $K_i \subset U$ is compact for each $i \in I$ and $K = \bigcup_{i\in I} K_i$, then the diagonal map $\delta : \Psi^K(U) \longrightarrow \prod_{i\in I} \Psi^{K_i}(U)$ is an embedding.*

Proof. We will show that $\Psi^K(U)$ has the initial topology w.r.t. the maps

$$\{\Psi^K(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)\}_{i\in I}.$$

It is a formal consequence of this, that δ is an embedding:

Since $\mathrm{id}^{\mathrm{s}} : \Psi^K(U) \longrightarrow \Psi^{K_i}(U)$ is injective, δ is injective, and so to show that δ is an embedding we simply need to show that the left-inverse δ^{-1} of $\delta|^{\mathrm{im}\,\delta}$ is continuous. The function $\delta^{-1} : \mathrm{im}\,\delta \longrightarrow \Psi^K(U)$ is continuous if and only if the compositions

$$\mathrm{im}\,\delta \xrightarrow{\delta^{-1}} \Psi^K(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)$$

are continuous for all $i \in I$, and this is the case, since they are just the i^{th} projection restricted to $\mathrm{im}\,\delta$.

We will now show $\Psi^K(U)$ has the initial topology w.r.t. $\{\Psi^K(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)\}_{i\in I}$: It is clear (once one notes, it is well-defined) that the diagram

$$(\Psi^{\mathrm{s}}|_U)_K \xrightarrow{((-)_{K_i\subset K})_{i\in I}} \prod_{i\in I} (\Psi^{\mathrm{s}}|_U)_{K_i} \underset{b}{\overset{a}{\rightrightarrows}} \prod_{(i,j)\in I\times I} (\Psi^{\mathrm{s}}|_U)_{K_i\cap K_j} \tag{C.2}$$

in **Top** forgets to an equalizer diagram in **Set**, where a, b are the unique maps such that

$$\mathrm{pr}_{(i,j)} \circ a = (\mathrm{pr}_i(-))|^{U_i}_{U_i\cap U_j}$$
$$\mathrm{pr}_{(i,j)} \circ b = (\mathrm{pr}_j(-))|^{U_j}_{U_i\cap U_j}.$$

Therefore, we have for any $A \subset (\Psi^{\mathrm{s}}|_U)_K$ that

$$((-)_{K_i\subset K})_{i\in I}(A) = \mathrm{im}\,\delta \cap \bigcap_{i\in I} \mathrm{pr}_i^{-1}((-)_{K_i\subset K}(A)). \tag{C.3}$$

Since the map $\pi_L \colon \Psi^{\mathrm{cs}}(U) \longrightarrow (\Psi^{\mathrm{s}}|_U)_L$ is surjective and open for all compact $L \subset U$, it follows from commutativity of

$$\begin{array}{ccc} (\Psi^{\mathrm{s}}|_U)_K & \xleftarrow{\pi_K} & \Psi^{\mathrm{cs}}(U) \\ {\scriptstyle (-)_{K_i \subset K}}\downarrow & \swarrow {\scriptstyle \pi_{K_i}} & \\ (\Psi^{\mathrm{s}}|_U)_{K_i}, & & \end{array}$$

that $(-)_{K_i \subset K}$ is open for all $i \in I$. Since each $(-)_{K_i \subset K}$ is open, it follows from (C.3) that $((-)_{K_i \subset K})_{i \in I}$ is relatively open (here we use that I is finite). The map $((-)_{K_i \subset K})_{i \in I}$ is injective, because (C.2) forgets to an equalizer diagram in **Set**, and so $((-)_{K_i \subset K})_{i \in I}$ is an embedding. Since $((-)_{K_i \subset K})_{i \in I}$ is an embedding $\Psi^K(U)$ has the initial topology w.r.t. $((-)_{K_i \subset K})_{i \in I}$ (and the diagram forms an equalizer diagram in **Top**). Since the product topology is initial w.r.t. the projections, we have that $\Psi^K(U)$ has the initial topology w.r.t. $\{(-)_{K_i \subset K}\}_{i \in I}$. Now, it follows from the commutative diagram

$$\begin{array}{ccc} \Psi^K(U) & \xrightarrow{\mathrm{id}^{\mathrm{s}}} & \Psi^{K_i}(U) \\ {\scriptstyle \text{initial}}\downarrow{\scriptstyle \pi_K} & & {\scriptstyle \pi_{K_i}}\downarrow{\scriptstyle \text{initial}} \\ (\Psi^{\mathrm{s}}|_U)_K & \xrightarrow[(-)_{K_i \subset K}]{} & (\Psi^{\mathrm{s}}|_U)_{K_i} \end{array}$$

for each $i \in I$ and left-cancel-ability of being initial w.r.t., that $\Psi^K(U)$ has the initial topology w.r.t. the maps $\{\Psi^K(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)\}_{i \in I}$. $\square$

Theorem C.2.7. *The presheaf Ψ is a sheaf.*

Proof. We have to show that Ψ lifts the sheaf condition of Ψ^{s} to **Top**. Let U be an open subset of $\mathbb{R}^n$ and let $\{U_i\}_{i \in I}$ be an open cover of U. We have to show, that

$$\Psi(U) \xrightarrow{\ e\ } \prod_{i \in I} \Psi(U_i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \Psi(U_i \cap U_j) \tag{C.4}$$

is an equalizer diagram in **Top**, where the maps are defined as in definition B.2.1. We do this by showing it exhibits the universal property:

Let X be a topological space and let $f \colon X \longrightarrow \prod_{i \in I} \Psi(U_i)$ be a continuous map such that $af = bf$. Since Ψ^{s} is a sheaf of sets, there is a unique function $\tilde{f}$ such that $f = e\tilde{f}$. Showing that (C.4) exhibits the universal property amounts to showing that $\tilde{f}$ is continuous. By definition of $\Psi(U)$, the function $\tilde{f}$ is continuous if and only if the composition

$$X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U)$$

is continuous for all compact subsets $K \subset U$. Let $K \subset U$ be a compact subset. Use that K is compact, to pick a finite sub-cover $\{V_i\}_{i \in J}$ of $\{U_i\}_{i \in I}$ of K and a Lebesgue number ϵ of this cover. Now define

$$K_i := \{x \in K \cap V_i \mid \operatorname{dist}(x, \overline{K \cap V_i} \backslash K) \geq \epsilon\}$$

for all $i \in J$, and note that $K = \bigcup_{i \in J} K_i$, and that K_i is compact for all $i \in J$. Since the diagonal δ is an embedding (lemma C.2.6) we have that

$$X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U)$$

is continuous if and only if

$$X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U) \xrightarrow{\delta} \prod_{j \in J} \Psi^{K_j}(U)$$

is continuous, which is continuous if and only if

$$X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U) \xrightarrow{\delta} \prod_{j\in J} \Psi^{K_j}(U) \xrightarrow{\mathrm{pr}_i} \Psi^{K_i}(U) \tag{C.5}$$

is continuous for all $i \in J$. Let $i \in J$ and note that (C.5) equals

$$X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)$$

and, since ρ is an embedding (lemma C.2.4), we have that $X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U)$ is continuous if and only if $X \xrightarrow{\tilde{f}} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^{K_i}(U) \xrightarrow{\rho} \Psi^{K_i}(V_i)$ is continuous. Using commutativity of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & \Psi(U) & \xrightarrow{\mathrm{id}^{\mathrm{s}}} & \Psi^{K_i}(U) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle e} & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \rho} \\ \prod_{j\in I} \Psi(U_j) & \xrightarrow[\mathrm{pr}_i]{} & \Psi(V_i) & \xrightarrow[\mathrm{id}^{\mathrm{s}}]{} & \Psi^{K_i}(V_i), \end{array}$$

we see that this is the case, finishing the proof. □

Theorem C.2.8. *If U and V are open subsets of $\mathbb{R}^n$, then the function*

$$p\colon \mathrm{Emb}(U,V) \times \Psi(V) \longrightarrow \Psi(U)$$

defined by $(j, M) \mapsto j^{-1}(M)$ is well-defined and continuous.

Proof (sketch). The function p is clearly well-defined.

The function p is continuous if and only if the compositions

$$\mathrm{Emb}(U,V) \times \Psi(V) \xrightarrow{p} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U)$$

are continuous for every compact $K \subset U$, and each of these is continuous if and only if the composition

$$\mathrm{Emb}(U,V) \times \Psi(V) \xrightarrow{p} \Psi(U) \xrightarrow{\mathrm{id}^{\mathrm{s}}} \Psi^K(U) \xrightarrow{\pi_K} (\Psi^{\mathrm{s}}|_U)_K$$

is continuous. We will now show this is the case for any given compact subset $K \subset U$. We will check the continuity locally. Let $\mathrm{Diff}_{\mathrm{c}}(U)$ denote the subspace of $\mathrm{Diff}(U)$ (with the compact-open topology) consisting of those f for which $f - \mathrm{id}_U$ is compactly supported. Let $j_0 \in \mathrm{Emb}(U,V)$. Pick a map $\lambda \in C^\infty_{\mathrm{c}}(U, [0,1])$ with $K \subset \mathrm{int}(\lambda^{-1}(1))$. Let

$$\phi\colon \mathrm{Emb}(U,V) \dashrightarrow \mathrm{Diff}_{\mathrm{c}}(U)$$

be the partially defined map given by

$$\phi(j)(x) = (1-\lambda(x))x + \lambda(x)\cdot j_0^{-1}(j(x))$$

for all j in a sufficiently small neighbourhood of j_0.

Let U_0 denote an open neighbourhood of $j_0 \in \mathrm{Emb}(U,V)$ on which ϕ is defined. Consider the diagram

$$\begin{array}{ccccccc} U_0 \times \Psi(V) & \hookrightarrow & \mathrm{Emb}(U,V)\times\Psi(V) & \xrightarrow{p} & \Psi(U) & \xrightarrow{\mathrm{id}^{\mathrm{s}}} & \Psi^K(U) \\ {\scriptstyle \phi\times j_0^{-1}(-)}\downarrow & & & & & & \downarrow{\scriptstyle \pi_K} \\ \mathrm{Diff}_{\mathrm{c}}(U)\times\Psi(U) & & \xrightarrow[\alpha]{} & & \Psi(U) & \xrightarrow[\pi_K]{} & (\Psi^{\mathrm{s}}|_U)_K, \end{array}$$

where α is the right action of $\mathrm{Diff}_{\mathrm{c}}(U)$ on $\Psi(U)$ given by precomposition. Now note that $j(x) = j_0(\phi(j)(x))$ for all $j \in U_0$ and all x in a sufficiently small neighbourhood of K, by definition of ϕ, so the diagram is commutative.

We will now sketch why ϕ, $j_0^{-1}(-)$, and α are continuous. Continuity of ϕ follows from general properties of the compact-open topology. Continuity of $j_0^{-1}(-)$ follows from $j_0^{-1}(-)\colon \Psi^{\mathrm{cs}}(V) \longrightarrow \Psi^{\mathrm{cs}}(U)$ being continuous (which can be checked using our subbasis for $\Gamma_{\mathrm{c}}(\mathbb{R}^n)$ and the open embeddings c_M) and commutativity of the diagram

$$\begin{array}{ccccccc}
\Psi(V) & \xrightarrow{\mathrm{id}^{\mathrm{s}}} & \Psi^K(V) & \xrightarrow{\pi_K} & (\Psi^{\mathrm{s}}|_V)_K & \xleftarrow{\pi_K} & \Psi^{\mathrm{cs}}(V) \\
{\scriptstyle j_0^{-1}(-)}\downarrow & & {\scriptstyle j_0^{-1}(-)}\downarrow & & \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle j_0^{-1}(-)} \\
\Psi(U) & \xrightarrow{\mathrm{id}^{\mathrm{s}}} & \Psi^K(U) & \xrightarrow{\pi_K} & (\Psi^{\mathrm{s}}|_U)_K & \xleftarrow{\pi_K} & \Psi^{\mathrm{cs}}(U),
\end{array}$$

where J is the unique function making the diagram commute, for all compact $K \subset V$. Continuity of α is proved analogously to $j_0^{-1}(-)$.

Since ϕ and $j_0^{-1}(-)$ are continuous, it follows that p is continuous when restricted to $U_0 \times \Psi(U)$ and thus p is continuous. $\square$

Appendix D. Some theory from GMTW

In this appendix we define some of the technical tools of [3] and [6]. The author originally attempted to read those before switching to [4] and wants to include some interesting points he learned when reading those. so, all the theory in this appendix comes from those.

We generalize $\mathscr{C}$-valued sheaves on topological spaces to $\mathscr{C}$-valued sheaves on the category $\mathscr{X}$. Then we introduce concordance on - and geometric realization of - sheaves of sets on $\mathscr{X}$ and note, that the latter represents the former. Concordance gives us a criterion for checking, on the sheaf level, whether the realization of a morphism of sheaves of sets on $\mathscr{X}$ is a weak homotopy equivalence. This criterion plays a major rôle in [3].

Warning. The contents of this appendix are almost completely irrelevant for the main sections of this thesis and their quality may be spotty.

Definition D.0.1. We let $\mathscr{X}$ denote the category of smooth, finite-dimensional manifolds without boundary and smooth maps between them. ⌟

D.1. Sheaves on $\mathscr{X}$ with values in a complete category

Definition D.1.1. Let $\mathscr{C}$ be a complete category. A *$\mathscr{C}$-valued sheaf on $\mathscr{X}$* is a contravariant functor $\mathscr{F}\colon \mathscr{X}^{\mathrm{op}} \longrightarrow \mathscr{C}$ such that the composition

$$X_{\mathrm{Zar}}^{\mathrm{op}} \xhookrightarrow{\iota^{\mathrm{op}}} \mathscr{X}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathscr{C}$$

is a $\mathscr{C}$-valued sheaf on X for all $X \in \mathscr{X}$, where $\iota\colon X_{\mathrm{Zar}} \hookrightarrow \mathscr{X}$ denotes the (non-full) inclusion. For a smooth map f in $\mathscr{X}$ we denote $\mathscr{F}(f)$ by f^* suppressing $\mathscr{F}$. The category of $\mathscr{C}$-valued sheaves on $\mathscr{X}$ is the full subcategory $\mathrm{Sh}(\mathscr{X};\mathscr{C}) \hookrightarrow \mathrm{PSh}(\mathscr{X};\mathscr{C})$ spanned by $\mathscr{C}$-valued sheaves on $\mathscr{X}$. ⌟

Definition D.1.2. If $\mathscr{F} \in \mathrm{Sh}(\mathscr{X};\mathscr{C})$, $X \in \mathscr{X}$, and $A \subset X$ we denote the stalk of $X_{\mathrm{Zar}}^{\mathrm{op}} \hookrightarrow \mathscr{X}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathscr{C}$ (if it exists) by $\mathscr{F}_A$, suppressing X. ⌟

D.2. Concordance

In this subsection, we will look at **Set**-valued sheaves on $\mathscr{X}$.

Note that the extended (or open) simplices

$$\Delta_{\mathrm{e}}^n \coloneqq \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \cdots + t_n = 1\}$$

form a cosimplicial object in $\mathscr{X}$ (with the evident face- and degeneracy-maps) which we will denote by $\boldsymbol{\Delta}_{\mathrm{e}}\colon \boldsymbol{\Delta} \longrightarrow \mathscr{X}$.

Definition D.2.1. Let $\mathscr{F} \in \mathrm{Sh}(\mathscr{X};\mathbf{Set})$. The *geometric realization* of $\mathscr{F}$, is the geometric realization of the simplicial set given by the composition

$$\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{\boldsymbol{\Delta}_{\mathrm{e}}^{\mathrm{op}}} \mathscr{X}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathbf{Set}\,.$$

Clearly, geometric realization forms a functor $|-|_{\mathrm{s}}\colon \mathrm{Sh}(\mathscr{X};\mathbf{Set}) \longrightarrow \mathbf{Top}$. We denote the geometric realization of $\mathscr{F}$ by $|\mathscr{F}|_{\mathrm{s}}$. ⌟

This directly motivates the following definition.

Definition D.2.2. We say that a map $\mathscr{F} \longrightarrow \mathscr{G}$ in $\mathrm{Sh}(\mathscr{X};\mathbf{Set})$ is a *weak equivalence* if the induced map $|\mathscr{F}|_{\mathrm{s}} \longrightarrow |\mathscr{G}|_{\mathrm{s}}$ is a weak homotopy equivalence. ⌟

We will now define a functor $[-]\colon \operatorname{Sh}(\mathscr{X};\mathbf{Set}) \longrightarrow \operatorname{PSh}(\mathscr{X};\mathbf{Set})$ that geometric realization will represent (in the sense of theorem D.2.7) called *concordance*.

Definition D.2.3. Let $\mathscr{F} \in \operatorname{Sh}(\mathscr{X};\mathbf{Set})$ and $X \in \mathscr{X}$. We define the *concordance* relation $\sim$ on $\mathscr{F}(X)$ by letting $t_0 \sim t_1$ for $t_0, t_1 \in \mathscr{F}(X)$, if there exists $s \in \mathscr{F}(X \times \mathbb{R})$ such that

$$s_{X\times(-\infty,0]} = (\operatorname{pr}^*(t_0))_{X\times(-\infty,0]} \quad \text{and} \quad s_{X\times[1,\infty)} = (\operatorname{pr}^*(t_1))_{X\times[1,\infty)}.$$

In this case we say that t_0 and t_1 are *concordant*, that s is a *concordance* from t_0 to t_1, and we may write $t_0 \sim_s t_1$. ⌟

Though the following is a rather routine check – and might very well be apparent to the reader – we include it for thoroughness anyway.

Proposition D.2.4. *If $\mathscr{F} \in \operatorname{Sh}(X;\mathbf{Set})$ and $X \in \mathscr{X}$, then concordance is an equivalence relation on $\mathscr{F}(X)$. Denoting the set of concordance classes of $\mathscr{F}(X)$ by $\mathscr{F}[X]$, this assembles to a functor $[-]\colon \operatorname{Sh}(\mathscr{X};\mathbf{Set}) \longrightarrow \operatorname{PSh}(\mathscr{X};\mathbf{Set})$.*

Proof. For reflexivity, we note that pr^*t is a concordance $t \sim t$ for all $t \in \mathscr{F}(X)$. Let $r_{1/2}\colon \mathbb{R} \longrightarrow \mathbb{R}$ denote the diffeomorphism defined by reflecting about $1/2 \in \mathbb{R}$. If $t_0 \sim_s t_1$ then $t_1 \sim_{r^*_{1/2}(s)} t_0$ which gives symmetry.

For transitivity, assume $t_0 \sim_{s_0} t_1$ and $t_1 \sim_{s_1} t_2$. Pick smooth, increasing maps $\phi, \psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\phi(x) = \begin{cases} x, & x \leq 0 \\ x - 2/3, & x \geq 1 \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} x + 2/3, & x \leq 0 \\ x, & x \geq 1. \end{cases}$$

Define

$$\mathfrak{s}_0 := (\operatorname{id}_X \times \phi|^{(-\infty,2/3)}_{(-\infty,4/3)})^*(s_0|_{(-\infty,4/3)}) \in \mathscr{F}(X \times (-\infty, 2/3)) \quad \text{and}$$
$$\mathfrak{s}_1 := (\operatorname{id}_X \times \psi|^{(1/3,\infty)}_{(-1/3,\infty)})^*(s_1|_{(-1/3,\infty)}) \in \mathscr{F}(X \times (1/3, \infty)).$$

Note that

$$\mathfrak{s}_0|_{X\times(1/3,2/3)} = \operatorname{pr}^*t_1 = \mathfrak{s}_1|_{X\times(1/3,2/3)},$$

so by the sheaf property there exists a unique $\mathfrak{s} \in \mathscr{F}(X \times \mathbb{R})$ such that

$$\mathfrak{s}|_{X\times(-\infty,2/3)} = \mathfrak{s}_0 \quad \text{and} \quad \mathfrak{s}|_{X\times(1/3,\infty)} = \mathfrak{s}_1.$$

It now follows, that $t_0 \sim_{\mathfrak{s}} t_2$. □

Warning. If $\mathscr{F} \in \operatorname{Sh}(\mathscr{X};\mathbf{Set})$, then $\mathscr{F}[-]$ is in general *not* a sheaf.

We will need a relative version of concordance for our results.

For $\mathscr{F} \in \operatorname{Sh}(\mathscr{X};\mathbf{Set})$ and $A \subset X$ a closed subset, note that the projection $X \times \mathbb{R} \longrightarrow X$ induces a function $\operatorname{pr}^*_A\colon \mathscr{F}_A \longrightarrow \mathscr{F}_{A\times\mathbb{R}}$ defined by

$$[(U, s)] \overset{\operatorname{pr}^*_A}{\longmapsto} [(U \times \mathbb{R}, \operatorname{pr}^*s)].$$

Definition D.2.5. Let $\mathscr{F} \in \operatorname{Sh}(\mathscr{X};\mathbf{Set})$, $X \in \mathscr{X}$, $A \subset X$ be a closed subset and $s \in \mathscr{F}_A$. Let $\mathscr{F}(X, A; s)$ denote the set $\{t \in \mathscr{F}(X) \mid t_A = s\}$. We define the *relative concordance* relation $\sim$ rel. A on $\mathscr{F}(X, A; s)$ by letting $t_0 \sim t_1$ rel. A if there exists $s' \in \mathscr{F}(X \times \mathbb{R})$ such that

$$t_0 \sim_{s'} t_1 \quad \text{and} \quad s'_{A\times\mathbb{R}} = \operatorname{pr}^*_A(s).$$

In this case we say that t_0 and t_1 are *concordant relative to* A, that s' is a *concordance relative to* A from t_0 to t_1, we may write $t_0 \sim_{s'} t_1$ rel. A, and we denote the set of relative concordance classes $\mathscr{F}[X, A; s]$. ⌟

Lemma D.2.6. *A map $\tau\colon \mathscr{F} \longrightarrow \mathscr{G}$ in $\mathrm{Sh}(\mathscr{X};\mathbf{Set})$ is a weak equivalence if the induced map*

$$\mathscr{F}[X,A;s] \longrightarrow \mathscr{G}[X,A;\tau(s)]$$

is surjective for all (X,A,s) as in definition D.2.5.

Proof. This is [6, Lemma 2.4.4]. □

Theorem D.2.7. *If $\mathscr{F} \in \mathrm{Sh}(\mathscr{X};\mathbf{Set})$, $x_0 \in X \in \mathscr{X}$, $A \subset X$ a closed subset of X, and $s \in \mathscr{F}_A$, then we have a bijection*

$$\mathscr{F}[X] \cong [X, |\mathscr{F}|_{\mathrm{s}}],$$

natural in X. More generally we have a natural bijection

$$\mathscr{F}[X,A;s] \cong [(X,A),(|\mathscr{F}|_{\mathrm{s}},s)].$$

Proof. See appendix A of [6]. □

D.3. Realising a sheaf of categories on $\mathscr{X}$

Warning. In appendix D.2, we defined $|\mathscr{F}|_{\mathrm{s}}$ for a sheaf of *sets* on $\mathscr{X}$ and this was a topological *space*. We will now define $|\mathscr{F}|_{\mathrm{c}}$ for a sheaf of *categories* and this will be a topological *category*. In the literature these are both denoted $|\mathscr{F}|$, which can cause confusion.

Definition D.3.1. We note, that we have an isomorphism of categories

$$\mathrm{Sh}(\mathscr{X};\mathbf{Cat}) = \mathrm{Sh}(\mathscr{X};\mathbf{Cat}(\mathbf{Set})) \cong \mathbf{Cat}(\mathrm{Sh}(\mathscr{X};\mathbf{Set})).$$

Realizing a **Set**-valued sheaf on $\mathscr{X}$ commutes with finite limits, since geometric realization of simplicial sets commutes with finite limits, and limits are calculated point-wise in functor categories. Thus, the realization functor induces a functor $|{-}|_{\mathrm{s}}\colon \mathbf{Cat}(\mathrm{Sh}(\mathscr{X};\mathbf{Set})) \longrightarrow \mathbf{Cat}(\mathbf{Top})$. We let $|{-}|_{\mathrm{c}}$ denote the composition

$$|{-}|_{\mathrm{c}}\colon \mathrm{Sh}(\mathscr{X};\mathbf{Cat}) \cong \mathbf{Cat}(\mathrm{Sh}(\mathscr{X};\mathbf{Set})) \xrightarrow{|-|_{\mathrm{s}}} \mathbf{Cat}(\mathbf{Top})$$

and call it *realization* of a **Cat**-valued sheaf on $\mathscr{X}$. ⌟

Remark. Writing it out, we have that

$$\mathrm{ob}\,|\mathscr{F}|_{\mathrm{c}} = |N_0\mathscr{F}|_{\mathrm{s}} \quad \text{and} \quad \mathrm{mor}\,|\mathscr{F}|_{\mathrm{c}} = |N_1\mathscr{F}|_{\mathrm{s}}$$

which is well-defined since the nerve is a right-adjoint so it induces a functor

$$\mathrm{Sh}(\mathscr{X};\mathbf{Cat}) \xrightarrow{N} \mathrm{Sh}(\mathscr{X};\mathbf{sSet})$$

and limits are calculated point-wise, so we get $N_0\mathscr{F}, N_1\mathscr{F} \in \mathrm{Sh}(\mathscr{X};\mathbf{Set})$.

D.4. Modelling the classifying space at the sheaf level

Here we construct a functor $\beta\colon \mathrm{Sh}(\mathscr{X};\mathbf{Cat}) \longrightarrow \mathrm{Sh}(\mathscr{X};\mathbf{Set})$, which will form a model of the classifying space at the level of sheaves in the sense of theorem D.4.4. For this we will fix an infinite set J for indexing.

Definition D.4.1. Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite open cover of X. We define the topological category $X_{\mathcal{U}} \in \mathbf{Cat}(\mathbf{Top})$ by

$$\mathrm{ob}\,X_{\mathcal{U}} = \coprod_{\emptyset\neq R\subset I} X_R \quad \text{and} \quad \mathrm{mor}\,X_{\mathcal{U}} = \coprod_{\emptyset\neq R\subset I}\ \coprod_{R\subset S\subset I} X_S,$$

where X_R denotes $\bigcap_{i\in R} U_i$ for every $R \subset I$, with source and target morphisms defined on X_S indexed by $R \subset S$ by

$$d_1\colon X_S \xrightarrow{\mathrm{id}_{X_S}} X_S \hookrightarrow \coprod_{R\subset I} X_R \quad \text{and} \quad d_0\colon X_S \hookrightarrow X_R \hookrightarrow \coprod_{R\subset I} X_R.$$

The other structure maps are uniquely determined. ⌟

Remark. Unravelling the definition gives, that $X_{\mathcal{U}}$ is a topological poset with elements of the form (R, x), where $R \subset I$ and $x \in X_R$ and the partial order defined by

$$(S, x) \leq (R, y) \text{ if and only if } R \subset S \text{ and } x = y.$$

Remark. Note that an internal hom-functor is a right-adjoint and so induces a functor on internal categories: If $C \in \mathbf{Cat}(\mathbf{Top})$ satisfies that C_0 and C_1 are CGWH, where we have an internal hom, then $\underline{\mathrm{Hom}}(-, C)$, i.e. $\underline{\mathrm{Hom}}(-, C)_i := \underline{\mathrm{Hom}}(-, C_i)$ for $i = 0, 1$, is also a topological category.

Notation. For $X \in \mathscr{X}$ we let $\tilde{X}$ denote the sheaf $C^\infty(-, X)$ on $\mathscr{X}$.

Definition D.4.2. Let $\mathscr{F} \in \mathrm{Sh}(\mathscr{X}; \mathbf{Cat})$. Define $\beta\mathscr{F}$, as a presheaf of sets, on objects by

$$\beta\mathscr{F}(X) = \left\{ (\mathcal{U}, \Phi) \;\middle|\; \begin{array}{c} \mathcal{U} = \{U_j\}_{j \in J} \text{ a locally finite open cover of } X \\ \Phi\colon \tilde{X}_{\mathcal{U}} \longrightarrow \mathscr{F} \text{ in } \mathrm{Sh}(\mathscr{X}; \mathbf{Cat}) \end{array} \right\}$$

and on morphisms by pulling back. ⌟

Proposition D.4.3. *The construction of* β *in definition D.4.2 assembles to a functor* $\mathrm{Sh}(\mathscr{X}; \mathbf{Cat}) \longrightarrow \mathrm{Sh}(\mathscr{X}; \mathbf{Set})$.

Proof. This is stated beneath [6, def. 4.1.1]. The definition of β in [6] is worded slightly differently, but it is equivalent to definition D.4.2. □

Theorem D.4.4. *If* $\mathscr{F} \in \mathrm{Sh}(\mathscr{X}; \mathbf{Cat})$ *then* $|\beta\mathscr{F}|_{\mathrm{s}}$ *is weakly equivalent to* $B|\mathscr{F}|_{\mathrm{c}}$.

Very rough proof sketch. To show this it suffices to show that

$$\beta\mathscr{F}[X] \cong [X, B|\mathscr{F}|_{\mathrm{c}}]$$

naturally in X. Let $(\mathcal{U}, \Phi) \in \beta\mathscr{F}(X)$. A partition of unity $\{\lambda_j : j \in J\}$ subordinate to $\mathcal{U}$ defines a map from $\lambda\colon X \longrightarrow BX_{\mathcal{U}}$. The counit ϵ of $|-| \dashv \underline{\mathbf{Sing}}$ gives a continuous functor $\epsilon_X : |\underline{\mathbf{Sing}}(X_{\mathcal{U}})| = |\tilde{X}_{\mathcal{U}}|_c \longrightarrow X_{\mathcal{U}}$. Since ϵ_X induces a level-wise weak equivalence $N_k\epsilon_X : N_k|\tilde{X}_{\mathcal{U}}| \xrightarrow{\simeq} N_k X_{\mathcal{U}}$ we get a weak equivalence $BX_{\mathcal{U}} \simeq B|\tilde{X}_{\mathcal{U}}|$. So we get the composition

$$X \xrightarrow{\lambda} BX_{\mathcal{U}} \simeq B|\tilde{X}_{\mathcal{U}}| \xrightarrow{B|\Phi|} B|\mathscr{F}|$$

the homotopy class of which is unique up to concordance of $(\mathcal{U}, \Phi) \in \beta\mathscr{F}(X)$ and does not depend on the choice of partition of unity. This defines a map $\beta\mathscr{F}[X] \longrightarrow [X, B|\mathscr{F}|]$ and [3] claims that this is a bijection the proof of which is in [6, Appendix A]. □

References

[1] Glen E. Bredon. *Topology and Geometry*. Springer New York, 1993. ISBN: 9781475768480. (On p. 26.)

[2] Søren Galatius. "Stable homology of automorphism groups of free groups". In: *Annals of Mathematics* 173.2 (Mar. 2011). (On pp. 7, 8.)

[3] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss. "The Homotopy Type of the Cobordism Category". In: *Acta Mathematica* 202.2 (2009), pp. 195–239. (On pp. i, iii, 17, 33, 36.)

[4] Søren Galatius and Oscar Randal-Williams. "Monoids of moduli spaces of manifolds". In: *Geom. Topol.* 14.3 (May 2010), pp. 1243–1302. (On pp. i, iii, 1, 5, 11, 12, 15, 16, 25, 28, 33.)

[5] Saunders Mac Lane. *Categories for the working mathematician*. 2nd ed. Vol. 5. Graduate Texts in Mathematics. N.Y: Springer Science & Business Media, 1997. (On p. 20.)

[6] Ib Madsen and Michael Weiss. "The Stable Moduli Space of Riemann Surfaces: Mumford's Conjecture". In: *Annals of Mathematics* 165.3 (2007), pp. 843–941. (On pp. 33, 35, 36.)

[7] J. P. May. *A concise course in algebraic topology*. Chicago lectures in mathematics. Chicago, Ill: University of Chicago Press, 1999. (On p. 13.)

[8] J. P. May. *The Geometry of Iterated Loop Spaces*. Springer Berlin Heidelberg, 1972. (On p. 21.)

[9] Oscar Randal-Williams. "Embedded cobordism categories and spaces of submanifolds". en. In: *Int. Mathem. Res. Not.* (May 2010). (On p. 15.)

[10] Graeme Segal. "Categories and cohomology theories". en. In: *Topology* 13.3 (Sept. 1974), pp. 293–312. (On p. 19.)

[11] Nathalie Wahl. "Ribbon braids and related operads". PhD thesis. University of Oxford, 2001. (On pp. 15, 21.)