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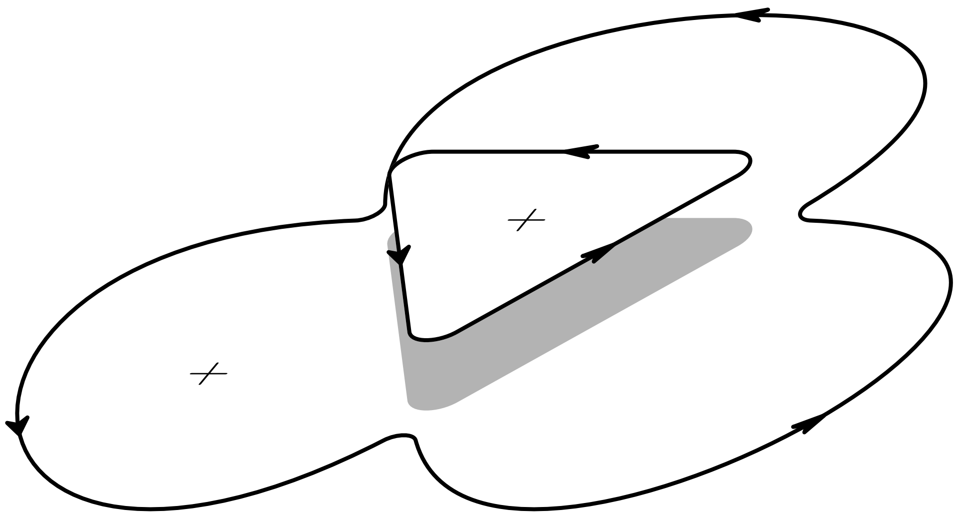

Project Outside Course Scope

Knot Theory

The Construction of the Alexander Polynomial

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KNOT THEORY

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ABSTRACT. We prove Alexander polynomial exists and is well-defined for smooth knots, black-boxing the most geometrically heavy details and assuming familiarity with basic algebraic topology i.e. homology, cohomology and covering maps. To do this, we introduce knots, Seifert surfaces, the linking pairing, deck transformations, the Alexander module, finitely presented modules and the order ideal.

CONTENTS

Introduction

Knot theory is a fascinating branch of mathematics, whose pioneers include many of the big names in mathematics. It has recently had a large resurgence as it has found many applications and is, as of writing this, a very active area of research. The Alexander polynomial, discovered by James Waddell Alexander II in 1923, was the first knot polynomial (of many). In this project we will introduce the necessary theory to define the Alexander polynomial and prove its existence for smooth knots assuming familiarity with basic algebraic topology and some manifold theory.

Structure of this project. In section 1 we give our working definition of *knot* (namely smooth knots), mention some knot invariants (like the knot complement) and define Seifert surfaces.

In section 2 we first recall some covering space theory, specifically on deck transformations, and use these to define the knot invariant called the *Alexander module*.

In section 3 we describe the Seifert algorithm, giving a Seifert surface for a smooth knot. We also state *Alexander duality* and use this to give a sketch of another proof for the existence of Seifert surfaces for any smooth codimension 2 knot. Then we introduce the *linking pairing* and *Seifert matrices*, which we need for our construction of the Alexander polynomial – the main focus of this project.

Section 4 is the heart of this project. Here, we first introduce finite presentability of modules and the order ideal and then we give a proof that the Alexander module of a knot is finitely presented with a principal order ideal. This ensures, that the Alexander polynomial exists.

Prerequisites. We will assume the reader is familiar with the basics of the language of categories and *very* basic commutative algebra. On the algebraic topology side, we will only use *very* basic theory for the most part, though, in the sketch of the proof of the existence of Seifert surfaces, we will use results from [4].

For readers familiar with the courses at the University of Copenhagen: We will basically only use theory from AlgTopI and Geom2, but at times we will draw upon AlgTopII, KomAlg and CatTop.

1. Knots, first steps

Here we will take the first steps by defining knots and providing some of the essential tools for working with knots.

1.1. Knots & Links

The objects of study of this project are *knots*. In the literature, there are a plethora of (non-equivalent) definitions of "knot". We will begin by giving a definition of a *topological knot*:

Definition 1.1.1. A *topological knot* is a topological embedding $K\colon S^n \hookrightarrow X$ of a sphere S^n into a topological space X. We say that two topological knots $K, K'\colon S^n \hookrightarrow X$ are *equivalent* if there exists a homeomorphism $h\colon X \longrightarrow X$ such that $K' = h \circ K$. In this case, we call h an equivalence between K and K'. ∘

Warning. This will *not* be our working definition of a knot.

The reason we will not focus on topological knots, is that they can be *wild* (read: infinitely complicated, e.g. space-filling). To avoid wild knots, one can put niceness restrictions on the knots e.g. requiring X to be a topological (resp. smooth or piece-wise linear) manifold, the embedding be topologically locally flat (resp. smooth or piece-wise linear) and the equivalences be topologically locally flat (resp. smooth or piece-wise linear or ambient isotopies). For added niceness one can also add orientations to X and S^n and require h to respect them. This leads us to our working definition of a knot:

Definition 1.1.2 (Knot)**.** Let X be a smooth manifold. A (smooth) knot in X is an oriented smooth embedding $K\colon S^n \hookrightarrow X$. Two knots $K, K'\colon S^n \hookrightarrow X$ are equivalent if there exists a diffeomorphism $h\colon X \longrightarrow X$ such that h respects the orientation and $K' = h \circ K$. ∘

Even though we will focus on the smooth case, a lot of our results work even in the topological case. We *highly* recommend the reader consider which cases the results presented in this project apply to.

We will mainly focus on the case $X = S^{n+2}$ and usually $n = 1$. The reason we direct our focus on codimension 2 knots is that, in the piece-wise linear and topologically locally flat case, there is only one knot of codimension greater than 2. Furthermore, to quote Rolfsen [7, p. 8]:

> ... knot theory, as well as link theory, is also more-or-less trivial in codimension one.

We would be amiss if we did not, in a project on knot theory, at least mention *links*. What could be better that one knot? Multiple knots! In definition 1.1.2, replacing S^n by $\coprod_I S^{k_i}$, giving I an order and requiring h respect that order gives the definition of a *link* and equivalences thereof.

1.2. First Steps

In this section we introduce some of the most basic tools of knot theory not all of which will be relevant for this project (but we feel an introduction to knot theory should at least mention these – though, in our case, it messes with the structuring).

1.2.1. *Seifert surfaces.* In this section we introduce *Seifert surfaces*, which are an essential tool in the study of knots (and links).

Definition 1.2.1. Let Y be a smooth manifold and $X \subset Y$ a submanifold. A smooth embedding $b\colon X \times [-1,1] \longrightarrow Y$ such that the restriction of b to $X \times \{0\}$ is the inclusion $X \hookrightarrow Y$ is called a *bicollar* of X (in Y). If such a bicollar exists, X (and any embedding with image X) is said to be *bicollared* (in Y). ◦

Definition 1.2.2. Let $K\colon S^n \hookrightarrow S^{n+2}$ be a knot. A connected, bicollared, compact, smooth manifold $M^{n+1} \subset S^{n+2}$ such that $\partial M = \operatorname{im} K$ (with the correct orientation) is called a *Seifert surface* for K. ◦

Remark. Note that any Seifert surface is, in particular, orientable.

In section 3.1, we will sketch a proof that there exists a Seifert surface for each (smooth) knot $S^n \hookrightarrow S^{n+2}$.

Definition 1.2.3. Let $M^m \subset N^n$ be smooth manifolds. A smooth embedding $t\colon M \times D^{n-m} \longrightarrow N$ such that the restriction of t to $M \times \{0\}$ is the inclusion $M \hookrightarrow N$ is called a *tubular neighbourhood* of M. ◦

Remark. Note that a bicollar is just a special case of a tubular neighbourhood.

We will use the following lemma without proof:

Lemma 1.2.4. *Let $K\colon S^1 \hookrightarrow S^3$ be a knot. There exists a tubular neighbourhood $\nu(K)\colon (\operatorname{im} K) \times D^1 \longrightarrow S^3$ of K.*

Another obvious knot invariant is the following:

Definition 1.2.5. The *genus* of a knot K is the minimal possible genus of a Seifert surface for K. ◦

1.2.2. *Knot invariants.* Determining whether two knots are equivalent is non-trivial. *Knot invariants*, the hallmark of knot theory, help distinguish knots up to equivalence. One can describe knot invariants as functions from (a subset of) the set of knots to a set S sending equivalent knots to the same element of S.[1] The following are examples of knot invariants:

Definition 1.2.6. Let $K\colon S^n \hookrightarrow S^{n+2}$ be a knot.

i) For $n = 1$ the minimal number of self-intersections of a *regular projection*[2] of a knot equivalent to K, called the *crossing number* of K.

ii) The group $\pi_1(S^{n+2} \setminus \operatorname{im} K)$, called the *knot group* of K. ◦

These are only two invariants, but knot theory is *full* of them. The trade-off with knot invariants, are usually computability of the invariant vs. how sensitive it is, i.e. how "many" knots it can distinguish. As mentioned, the main result of this project is the construction of the knot invariant called the *Alexander polynomial.* Probably the most important knot invariant is the *knot complement*:

Notation. Let $K\colon S^n \hookrightarrow X$ be a knot. We will denote the complement $X \setminus \operatorname{im} K$ by $K^{\complement}$.

Since we focus on smooth knots, we will usually need to work with the following homotopic analogue of the knot complement:

Notation. Let $K\colon S^n \hookrightarrow X$ be a knot and $\nu(K)$ be a tubular neighbourhood of K. We will denote the complement $X \setminus \mathring{\nu}(K)$ by X_K (suppressing ν).

Remark. Note, that $K^{\complement} \simeq X_K$.

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[1] or more often, to *equivalent* elements of S if one has a notion of equivalence in S.

[2] see definition 3.1.1 – though it is what you think it is.

2. The Alexander Module

In this section we will construct the *Alexander module*, which is a knot invariant, that we will use in the construction of the Alexander polynomial. It is worth noting, that we actually remain in the *topological* case in this section (replacing X_K by simply $K^{\complement}$).

2.1. Reminder on Deck Transformations

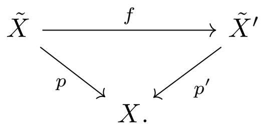
This section closely follows the exposition given in Chapter 1 of [5].

Definition 2.1.1. We call a covering, $p\colon \tilde{X} \longrightarrow X$, *regular* if $p_*\pi_1(\tilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, p(\tilde{x}))$ for all $\tilde{x} \in \tilde{X}$. ○

Definition 2.1.2. Let X be a topological space. Denote by $\mathrm{Cov}(X)$ the category with objects covering maps $p\colon \tilde{X} \longrightarrow X$ and morphisms

$$f\colon (p\colon \tilde{X} \longrightarrow X) \longrightarrow (p'\colon \tilde{X}' \longrightarrow X)$$

the morphisms $f\colon \tilde{X} \longrightarrow \tilde{X}'$ in **Top** such that the following triangle commutes



We will call $\mathrm{Cov}(X)$ the *category of coverings* of X. ○

We leave it to the reader to check that $\mathrm{Cov}(X)$ is a category for all $X \in$ **Top**.

Definition 2.1.3. Let $p\colon \tilde{X} \longrightarrow X$ be a covering map. A *deck transformation* of p is an isomorphism $p \cong p$ (in $\mathrm{Cov}(X)$). We denote by $\mathrm{Deck}(p)$ the group of deck transformations of p under composition i.e. $\mathrm{Deck}(p) \coloneqq \mathrm{Aut}_{\mathrm{Cov}(X)}(p)$. ○

Notation. For a group G we denote by $\mathcal{B}G$ the category with one object, $*$, and morphisms G.

Definition 2.1.4. Let G be a group and X a topological space. A (*continuous*) *action* of G on X is a functor $\mathcal{B}G \longrightarrow$ **Top** which sends $*$ to X. ○

For any covering map $p\colon \tilde{X} \longrightarrow X$ there is an obvious (continuous) action of the group $\mathrm{Deck}(p)$ on $\tilde{X}$, namely the action $\mathcal{B}\,\mathrm{Deck}(p) \longrightarrow$ **Top** sending $*$ to $\tilde{X}$ and every $f \in \mathrm{Deck}(p)$ to "itself" i.e. to $f\colon \tilde{X} \longrightarrow \tilde{X}$.

Lemma 2.1.5. *Let $X \in$ **Top** be a path-connected, locally path-connected space, let $x \in X$ and let $p, p' \in \mathrm{Cov}(X)$ be path-connected coverings of X. If $\tilde{x}$ and $\tilde{x}'$ are points in the fibres of p and p' over x respectively then there exists an isomorphism $f\colon p \longrightarrow p'$ taking $\tilde{x}$ to $\tilde{x}'$ if and only if the images of the induced maps of p and p' on fundamental groups are equal.*

Proof. Denote the coverings by $p\colon (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ and $p'\colon (\tilde{X}', \tilde{x}') \longrightarrow (X, x)$. If there exists an isomorphism $f\colon p \longrightarrow p'$ then $\operatorname{im} p_* = \operatorname{im} p'_*$ by [3, Theorem 2.7.2]. Conversely assume $\operatorname{im} p_* = \operatorname{im} p'_*$. Since $\operatorname{im} p_* = \operatorname{im} p'_*$ there exist unique lifts $\tilde{p}\colon \tilde{X} \longrightarrow \tilde{X}'$ and $\tilde{q}\colon \tilde{X}' \longrightarrow \tilde{X}$ of p and p' respectively (by e.g. [3, Theorem 2.7.2]). By uniqueness of these lifts we now have $\tilde{p} \circ \tilde{p}' = \mathrm{id}_{\tilde{X}'}$ and $\tilde{p}' \circ \tilde{p} = \mathrm{id}_{\tilde{X}}$ so $\tilde{p}$ is an isomorphism $p \longrightarrow p'$ such that $\tilde{p}(\tilde{x}) = \tilde{x}'$ and we are done. □

Lemma 2.1.6. *Let $p\colon (\tilde{X},\tilde{x}) \longrightarrow (X,x)$ be a path-connected covering and let $\tilde{x}'$ be a point in the fibre of p. Then the subgroup $p_*\pi_1(\tilde{X},\tilde{x}')$ is conjugate to $p_*\pi_1(\tilde{X},\tilde{x})$ in $\pi_1(X,x)$.*

Proof. Pick a path $\mu\colon \tilde{x} \rightsquigarrow \tilde{x}'$. The map $\mu_*\colon \pi_1(\tilde{X},\tilde{x}') \longrightarrow \pi_1(\tilde{X},\tilde{x})$ is an isomorphism. Since both $\tilde{x}$ and $\tilde{x}'$ are in the fibre of p, $p\circ\mu$ must be a loop at x. Let $[\lambda] \in \pi_1(\tilde{X},\tilde{x})$ and let $[\lambda']$ denote $(\mu_*)^{-1}([\lambda]) \in \pi_1(\tilde{X},\tilde{x}')$. Now

$$\begin{aligned} p_*[\lambda] &= p_*(\mu_*([\lambda'])) = p_*([\mu\cdot\lambda'\cdot\bar{\mu}]) = [p\circ(\mu\cdot\lambda'\cdot\bar{\mu})] = [p\circ\mu]\cdot[p\circ\lambda']\cdot[p\circ\bar{\mu}] \\ &= [p\circ\mu]\cdot p_*([\lambda'])\cdot[p\circ\mu]^{-1} \end{aligned}$$

and so, since p_* is injective, we have $p_*\pi_1(\tilde{X},\tilde{x}) = [p\circ\mu](p_*\pi_1(\tilde{X},\tilde{x}'))[p\circ\mu]^{-1}$. This proves the lemma. □

Lemma 2.1.7. *Deck transformations of path-connected coverings are uniquely determined by their image of any single point.*

Proof. Let $p\colon \tilde{X} \longrightarrow X$ be a path-connected covering, $\tilde{x} \in \tilde{X}$ and $f, g \in \mathrm{Deck}(p)$. Assume $f(\tilde{x}) = g(\tilde{x})$. We will now prove that $f = g$. Let $\tilde{x}' \in \tilde{X}$ and pick a path $\lambda\colon \tilde{x} \rightsquigarrow \tilde{x}'$. By definition of $\mathrm{Deck}(p)$ we have that $p\circ\lambda = p\circ f\circ\lambda = p\circ g\circ\lambda$ so both $f\circ\lambda$ and $g\circ\lambda$ are lifts of $p\circ\lambda$. Now, by uniqueness of lifts, we have $f\circ\lambda = g\circ\lambda$ so, in particular, $f(\tilde{x}') = f(\lambda(1)) = g(\lambda(1)) = g(\tilde{x}')$ and we are done. □

Remark. In particular, only $\mathrm{id} \in \mathrm{Deck}(p)$ has any fix-points for any path-connected covering p.

Proposition 2.1.8. *Let X be a path-connected and locally path-connected topological space and let $p\colon \tilde{X} \longrightarrow X$ be a path-connected, regular covering, then*

$$\mathrm{Deck}(p) \cong \pi_1(X, p(\tilde{x}))/p_*\pi_1(\tilde{X},\tilde{x})$$

for all $\tilde{x} \in \tilde{X}$.

Proof. Let $p\colon \tilde{X} \longrightarrow X$ be a regular covering and let $\tilde{x} \in \tilde{X}$. We will define the function $\phi\colon \pi_1(X,p(\tilde{x})) \longrightarrow \mathrm{Deck}(p)$ by $[\lambda] \mapsto \tau \in \mathrm{Deck}(p)$ defined by $\tau(\tilde{x}) = p(\tilde{x}).[\lambda]$ for all $[\lambda] \in \pi_1(X,p(\tilde{x}))$ and check that ϕ is well-defined. For ϕ to be well-defined we only need to check that such a τ always exists and is well-defined. Lemma 2.1.7 ensures that such a τ, if it exists, is well-defined and, because ϕ is regular, it follows from lemma 2.1.6 and lemma 2.1.5 that such a τ always exists, so ϕ is well-defined. The function ϕ is a homomorphism when restricting the deck transformations to the fibre of p, so by uniqueness ϕ is a homomorphism. Because p is regular, lemma 2.1.6 combined with lemma 2.1.5 gives us that ϕ is surjective. It now suffices to show, that $\ker\phi = p_*\pi_1(\tilde{X},\tilde{x})$. By lemma 2.1.7 we get that $\phi([\lambda]) = \mathrm{id}_{\tilde{X}}$ if and only if $p(\tilde{x}).[\lambda] = \tilde{x}$ i.e. if and only if λ lifts to a loop in $\tilde{X}$ along p, but by [3, corollary 2.2.10] this is the case if and only if $[\lambda] \in p_*\pi_1(\tilde{X},\tilde{x})$ so we are done. □

2.2. Defining the Alexander Module

In this section we will define the *Alexander module*, which is a knot invariant. We begin, by considering the following theorem (which is [3, theorem 9.4.2]):

Theorem 2.2.1. *For any $k, n \in \mathbb{N}_0$ and any injective map $i\colon S^k \hookrightarrow S^n$ we have*

$$\tilde{H}_*(S^n \backslash i(S^k)) \cong \begin{cases} \mathbb{Z} & \text{for } * = n-k-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Omitted. □

Vista. This is not just a coincidence. It is reminiscent of *Alexander duality.*

From the above theorem, it follows, that for a knot $K\colon S^n \hookrightarrow S^{n+2}$ the reduced homology of the complement X_K is

$$\tilde{H}_i(X_K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } i=1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular X_K is path-connected and $H_1(X_K;\mathbb{Z}) = \tilde{H}_1(X_K;\mathbb{Z}) \cong \mathbb{Z}$ and all higher homology groups are trivial, so only the first homology group is of interest.[3]

Definition 2.2.2. A path-connected covering $p\colon \tilde{X} \longrightarrow X$ is called an *infinite cyclic cover* (of X) if $\operatorname{Deck}(p) \cong \mathbb{Z}$. ∘

Remark. Note that for an infinite cyclic cover of a space X, there is a bijection between $\mathbb{Z}$ and the fibre of the covering (at any point).

To ensure the existence of the infinite cyclic cover of a knot-complement even for non-surjective *topological knots*, we will use the following point-set topological lemma, which will allow us to use the Galois correspondence for path-connected coverings.

Lemma 2.2.3. *For any non-surjective continuous map $K\colon S^k \longrightarrow S^n$, the complement $S^n \backslash K(S^k) \neq \emptyset$ is locally path-connected and semi-locally simply connected.*

Proof. Since S^k is compact so is $K(S^k)$ and, since S^n is Hausdorff, this means that $K(S^k)$ is closed in S^n and therefore $S^n \backslash K(S^k)$ is open in S^n. Pick a point $s \in S^k$. Now use stereographic projection to produce a homeomorphism $\phi\colon S^n \backslash \{K(s)\} \longrightarrow \mathbb{R}^n$. Using that S^n is Hausdorff we get that $\{K(s)\}$ is closed in S^n and therefore that $S^n \backslash \{K(s)\}$ is open in S^n. This gives us, that $S^n \backslash K(S^k)$ is an open subset of $S^n \backslash \{K(s)\}$. Since $S^n \backslash K(S^k)$ is an open subset of $S^n \backslash \{K(s)\}$ we can restrict ϕ to a homeomorphism from $S^n \backslash K(S^k)$ to an open (non-empty since K is not surjective) subset of $\mathbb{R}^n$. Open subsets of $\mathbb{R}^n$ are locally path-connected and locally contractible so they are *a fortiori* also semi-locally simply connected. □

Proposition 2.2.4. *If $K\colon S^n \hookrightarrow S^{n+2}$ is a (resp. non-surjective topological) knot, then there exists a unique (up to isomorphism) infinite cyclic covering of X_K (resp. $K^{\complement}$).*

Proof. Because X_K is sufficiently nice, we can use the classification of path-connected coverings to get the existence of a unique covering $p\colon \tilde{X}_K \longrightarrow X_K$ such that $p_*\pi_1(\tilde{X}_K, \tilde{x})$ is the commutator subgroup of $\pi_1(X_K, p(\tilde{x}))$ for all $\tilde{x} \in \tilde{X}_K$ (using that the commutator subgroup is normal). Now, since the commutator subgroup is normal, this covering is regular and so proposition 2.1.8 gives us, that

$$\operatorname{Deck}(p) \cong \pi_1(X,x)/\operatorname{im} p_* = \pi_1(X,x)^{\mathrm{ab}} \cong H_1(X_K;\mathbb{Z}) \cong \mathbb{Z}$$

for all $x \in X_K$ where we use theorem 2.2.1. So p is an infinite cyclic cover of X_K. □

Notation. We will denote the domain of the (path-connected) infinite cyclic cover of X_K by $\tilde{X}_K$ (which is, of course, also unique up to isomorphism).

Notation. We denote by Λ the group ring $\mathbb{Z}[\mathbb{Z}]$, which is isomorphic to $\mathbb{Z}[t^{\pm 1}]$ as a ring.

Lemma 2.2.5. *For any knot $K\colon S^n \hookrightarrow S^{n+2}$, the group $H_1(\tilde{X}_K;\mathbb{Z})$ has the structure of a Λ-module.*

[3] We will use later (without proof) that Alexander duality (theorem 3.1.3) also gives us, that the meridian of K is the generator of $H_1(X_K;\mathbb{Z})$.

Proof. Let $p\colon \tilde{X}_K \longrightarrow X_K$ denote the infinite cyclic cover of X_K. The (continuous) action $\mathcal{B}\,\mathrm{Deck}(p) \longrightarrow \mathbf{Top}$ of $\mathrm{Deck}(p)$ on $\tilde{X}_K$ gives us an action $\mathcal{B}\,\mathrm{Deck}(p) \longrightarrow \mathbf{Ab}$ of $\mathrm{Deck}(p)$ on $H_1(\tilde{X}_K;\mathbb{Z})$ by post-composition with $H_1(-;\mathbb{Z})\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$. Since $\mathrm{Deck}(p) \cong \mathbb{Z}$, this means we have an action of $\mathbb{Z}$ on $H_1(\tilde{X}_K;\mathbb{Z})$ and so, by lemma 2.2.8, $H_1(\tilde{X}_K;\mathbb{Z})$ has the structure of a $\mathbb{Z}[\mathbb{Z}]$-module (i.e. specifically the one given by using ϕ from construction 2.2.7). □

Definition 2.2.6. Let $K\colon S^n \hookrightarrow S^{n+2}$ be a knot. We will call $H_1(\tilde{X}_K;\mathbb{Z})$, with the Λ-module structure specified in the proof of lemma 2.2.5, the *Alexander Module* (of K) and denote it by $\mathcal{A}(K)$. ◦

Clearly the Alexander module is a knot invariant.

2.2.1. *Group rings.* This sections *raison d'être* is to explain (in an overly-complicated way) exactly how we got the $\mathbb{Z}[\mathbb{Z}]$-module structure on $\mathcal{A}(K)$.

Construction 2.2.7. Let G be a group and R be a commutative, unital ring. Define functors

$$\phi\colon\ \mathrm{Fun}(\mathcal{B}G, \mathbf{Mod}_R) \leftrightarrows \mathbf{Mod}_{R[G]} :\psi$$

by $\phi(F) = F(*)$ with the $R[G]$-module structure given by $mg = F(g)(m)$ for all $F\colon \mathcal{B}G \longrightarrow \mathbf{Mod}_R$ and $\phi(\alpha\colon F \Rightarrow H) = (\alpha_*\colon F(*) \longrightarrow H(*))$ which becomes $R[G]$-linear by definition of natural transformations.

Let $M \in \mathbf{Mod}_{R[G]}$ and define ψ by $\psi(M) = (F\colon \mathcal{B}G \longrightarrow \mathbf{Mod}_R)$ defined by $F(*) = U(M)$ where U is the restriction of scalars along the inclusion $R \hookrightarrow R[G]$ and $F(g)\colon U(M) \longrightarrow U(M)$ defined by $m \mapsto mg$.

Lemma 2.2.8. *Let G be a group and R be a commutative, unital ring, then the functors ϕ and ψ from construction 2.2.7 induce an equivalence of categories*

$$\phi\colon\ \mathrm{Fun}(\mathcal{B}G, \mathbf{Mod}_R) \cong \mathbf{Mod}_{R[G]} :\psi.$$

Proof. The proof amounts to unravelling the definitions. □

Remark. In particular, spelling this out, we can get: Let $A \in \mathbf{Mod}_R$ with R commutative and unital and let $G \in \mathbf{Grp}$. If G acts on A, then A, naturally, has the structure of an $R[G]$-module.

3. Seifert Surfaces

Having introduced Seifert surfaces in section 1.2, we will, in this section, sketch two proofs of their existence (in the smooth case) and use them to define the *linking number* of disjoint knots. We will then use these linking numbers to define *Seifert matrices* for a knot.

3.1. Existence of Seifert Surfaces

In this section we will describe the Seifert algorithm, which produces a Seifert surface for a knot and then we will sketch a more formal proof of the existence of Seifert surfaces for knots. This will also be one of the places in this project we restrict ourselves to the smooth category.

3.1.1. *Seifert algorithm.* In this section we will relax our demand for formality and rely more on geometric intuition due to the pain and opacity of the alternative.

We will use the following definition (inspired by [7, p. 63]) in the description of the algorithm:

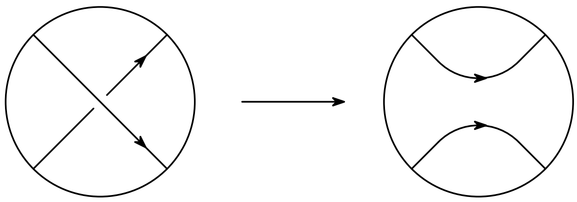
Definition 3.1.1. Let $K\colon S^1 \hookrightarrow \mathbb{R}^3$ be a knot and let $p\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the orthogonal projection onto a plane. If the composition $p \circ K\colon S^1 \longrightarrow \mathbb{R}^2$ never sends 3 distinct points to the same point and only intersects itself transversely, we call p a *regular projection* of K. ∘

We will use, without proof, that for any knot $S^1 \hookrightarrow \mathbb{R}^3$ there exists such a regular projection of an equivalent knot. The following algorithm is basically a reformulation of the proof in [7, p. 120]. We recommend the reader skim example 3.1.2 before reading the Seifert algorithm.

Algorithm (Seifert)**.** Let $K\colon S^1 \hookrightarrow \mathbb{R}^3$ be a (smooth) knot and fix a regular projection p of (a knot equivalent to) K. Let K' denote the image of p.

Step 1) Replace all crossings in K' as depicted in fig. 1.

Figure 1. Seifert algorithm step 1.



Now K' is a collection of disjoint Jordan-curves in the plane.

Step 2) Orient the discs bounded by the Jordan-curves of K', so that the boundary runs counterclockwise as seen from the "+" side.

Step 3) Offset each of the discs oriented in the previous step from the plane such that the innermost discs are on top.

Step 4) Connect the discs where the crossings were in the way depicted in fig. 2 producing a Seifert surface for K.

Example 3.1.2 (Seifert algorithm for the trefoil)**.** Taking $K\colon S^1 \hookrightarrow \mathbb{R}^3$ to be a trefoil, we illustrate the Seifert algorithm for K in fig. 3.

Figure 2. Seifert algorithm step 4.

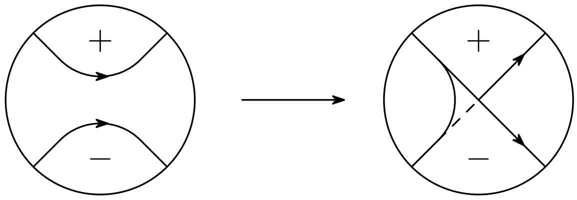
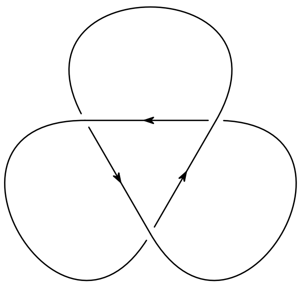
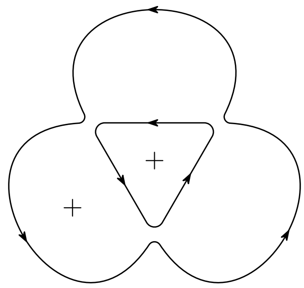


Figure 3. Seifert algorithm on the trefoil

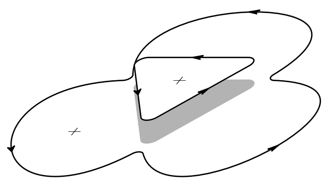
(a) Regular projection of a trefoil.



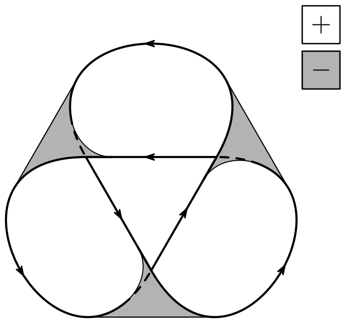
(b) Step 1 and 2.



(c) Step 3. The grey area is supposed to depict a *shadow*.



(d) Step 4. Here, the color is to differentiate between the "+" and "−" side.



3.1.2. *Alexander duality.* We will now sketch a proof of the existence of Seifert surfaces using Alexander duality. In the proof, we rely on multiple results we will not prove, including Alexander duality.

The proof uses a method for abstractly constructing a Seifert surface for any knot $S^n \hookrightarrow S^{n+2}$. Also, the method used to construct the Seifert surface yields *all* Seifert surfaces while the Seifert algorithm does not.

We will use the following formulation of Alexander duality from [5, Corollary 3.45], which this project, unfortunately, does not contain a proof of:

Theorem 3.1.3 (Alexander duality)**.** *If K is a locally contractible, compact, nonempty, proper subspace of S^n then*

$$\tilde{H}_i(K;\mathbb{Z}) \cong \tilde{H}^{n-i-1}(S^n \backslash K;\mathbb{Z})$$

for all $i \in \mathbb{Z}$.

Proof. Omitted. □

Theorem 3.1.4. *If $K\colon S^n \hookrightarrow S^{n+2}$ is a (smooth) knot, then there exists a Seifert surface for K.*

Sketch of proof. Let $K\colon S^n \hookrightarrow S^{n+2}$ be a knot. Pick a tubular neighbourhood $\nu(K)$ of K. Pick a non-zero element

$$[\gamma] \in H_1(\nu(K);\mathbb{Z}) \cong \mathbb{Z}, \quad [\gamma] \neq 0.$$

Note that $\nu(K)$ is a locally contractible, compact, nonempty, proper subspace of S^{n+2} so Alexander duality gives us, in particular, that

$$\tilde{H}_1(\nu(K);\mathbb{Z}) \cong \tilde{H}^1(X_K;\mathbb{Z}) = H^1(X_K;\mathbb{Z}) \cong [X_K, K(\mathbb{Z},1)] \cong [X_K, S^1],$$

where we have used that singular cohomology is represented by Eilenberg–MacLane spectra,[4] and that S^1 is a $K(\mathbb{Z},1)$. Sending $[\gamma]$ through the composition of these isomorphisms we thereby get a homotopy-class of maps $[f] \in [S^{n+2}\backslash\mathring{\nu}(K), S^1]$. Using smooth approximation, we may and do assume f is smooth. We may and do also assume $\operatorname{im} f$ has non-zero Lebesgue measure. Now, using Sards theorem [6, p. 10] we can pick a regular point of f in $\operatorname{im} f$ and assume WLOG that it is the point $1 \in S^1$. Using transversality we can assume f is transversal to 1. This now gives us, that $\Sigma \coloneqq f^{-1}(1)$ is a codimension 1 smooth manifold of S^{n+2} (by e.g. [2, Theorem 15.3]).

To make sure f behaves properly at $\partial(S^{n+2}\backslash\mathring{\nu}(K))$ we note that all the tools we have used have relative counterparts, so we could have done this entire construction relatively with $(X_K, \partial X_K)$. From doing this, we can get a codimension 1 smooth submanifold Σ of S^{n+2} with boundary K (orientation also follows from the construction). The smooth manifold Σ may not be path-connected, but we can throw away all path-components of Σ not containing K and get a Seifert surface for K. □

3.2. Linking Number

In this section we will define a link-invariant called the *linking number* (even for topological links).

Definition 3.2.1. Let $J, K\colon S^1 \hookrightarrow S^3$ be knots. We say that J and K are *disjoint* if $\operatorname{im} J$ and $\operatorname{im} K$ are disjoint. ∘

We now want to define the *linking number* of two disjoint knots. Since there are many equivalent definitions of the linking number we will give two different constructions, show that they give the same number (with a sign convention) and then define this to be the linking number. We only give two different constructions, but we highly recommend the reader take a look at [7, pp. 132–133], where Rolfsen gives *eight* equivalent definitions of the linking number – number seven of which being particularly entertaining and discovered by non other than Gauß.

Notation. We let $\hbar$ denote the Hurewicz homomorphism.

Construction 3.2.2. Let $J, K\colon S^1 \hookrightarrow S^3$ be disjoint topological knots.

- ℓk_1: Since J and K are disjoint, we have $\operatorname{im} J \subset K^{\complement}$ and so (since $H_1(K^{\complement};\mathbb{Z}) \cong \mathbb{Z}$ from theorem 2.2.1) we can pick a generator g of $H_1(K^{\complement};\mathbb{Z})$ and define $\ell k_1(J,K)$ to be the integer n such that $[J] = ng$, where $[J]$ denotes the homology class of J in $K^{\complement}$.
- ℓk_2: Fix an isomorphism $\phi\colon H_1(K^{\complement};\mathbb{Z}) \cong \mathbb{Z}$. Since J and K are disjoint, we have $\operatorname{im} J \subset K^{\complement}$ pick a point $x \in \operatorname{im} J$ we can define $\ell k_2(J,K) \coloneqq \phi(\hbar([J]))$ where $\hbar\colon \pi_1(K^{\complement}, x) \longrightarrow H_1(K^{\complement};\mathbb{Z})$ is the Hurewicz homomorphism.

Lemma 3.2.3. *The maps ℓk_1 and ℓk_2 in construction 3.2.2 are well-defined and agree up to sign, i.e. $\ell k_1(J,K) = \pm\ell k_2(J,K)$ for all disjoint knots $J, K\colon S^1 \hookrightarrow S^3$.*

[4] in the sense of [4, theorem 3.4.2] stating that $H^n(X;G) \cong [X, K(G,n)]$ for all $n \geq 0$.

Proof. Well-definedness of ℓk_2 is trivial. That $\ell k_1 = \pm \ell k_2$ is basically by definition; $\hbar$ is just the abelianization. We pick a sign-convention such that $\ell k_1 = \ell k_2$. Now well-definedness of ℓk_1 follows from $\ell k_1 = \ell k_2$. □

Definition 3.2.4. Let $J, K \colon S^1 \hookrightarrow S^3$ be disjoint knots. We define the *linking number* of J and K to be $\ell k_i(J,K)$ for $i = 1,2$ and denote it $\ell k(J,K)$. We immediately extend ℓk bilinearly to disjoint unions of simple curves, and get the *linking pairing* by noting that $\ell k(J,K) = \ell k(K,J)$ (from geometric intuition) and using ℓk_1 (which is already linear in one variable). ◦

3.3. Seifert Matrices

In this section we will introduce Seifert matrices following [7, p. 201]. A bicollar allows us to do the following:

Notation. Let $M \subset N$ be manifolds, b be a bicollar of M in N and let $x \colon X \longrightarrow M$ be a map. We denote by x^+ (suppressing b) the map $b \circ (x, 1) \colon X \longrightarrow N$ and likewise by x^- the map $b \circ (x, -1) \colon X \longrightarrow N$. We call x^+ the *pushoff* of x.

Definition 3.3.1. Let $K \colon S^1 \hookrightarrow S^3$ be a knot, Σ_g a Seifert surface for K and b a bicollar of Σ_g. We call a map $H_1(\mathring{\Sigma}_g;\mathbb{Z}) \times H_1(\mathring{\Sigma}_g;\mathbb{Z}) \longrightarrow \mathbb{Z}$ defined on 1-cycles $[x],[y]$ by
$$([x],[y]) \mapsto \ell k(x, y^+)$$
a *Seifert form* of K (this is well-defined, because we can fix one entry at a time). ◦

We will need the following lemma for our presentation of the Alexander module.

Lemma 3.3.2. *Let $K \colon S^1 \hookrightarrow S^3$ be a knot and Σ_g a Seifert surface for K. For all $[x],[y] \in H_1(\mathring{\Sigma}_g)$ we have:*
$$\ell k(x^-, y) = \ell k(x^-, y^+) = \ell k(x, y^+).$$

Proof. This follows easily, by using ℓk_1 and b to form a 1-boundary making $[y] = [y^+]$ in $H_1((x^-)^{\complement})$ and $[x] = [x^-]$ in $H_1((y^+)^{\complement})$. □

Definition 3.3.3. Let $f \colon H_1(\mathring{\Sigma}_g;\mathbb{Z}) \times H_1(\mathring{\Sigma}_g;\mathbb{Z}) \longrightarrow \mathbb{Z}$ be a Seifert form of K. Choosing a basis $([e_1],[e_2],\dots[e_{2g}])$ for $H_1(\mathring{\Sigma}_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$ we associate to f the matrix $V = (v_{i,j}) \in \mathrm{Mat}_{\mathbb{Z}}(2g,2g)$ defined by $v_{i,j} = \ell k(e_i, e_j^+)$. We call such a matrix V a *Seifert matrix* (for K). ◦

Remark. A Seifert matrix is not a *knot* invariant. It depends on the choice of Seifert surface and bicollar.

Warning. For a fixed Seifert form f, Seifert matrices are only preserved up to congruence: Let V be a Seifert matrix for f. A matrix W over $\mathbb{Z}$ is a Seifert matrix for f if and only if $W = PVP^\dagger$ for some invertible matrix P over $\mathbb{Z}$.

Lemma 3.3.4. *Let $K \colon S^1 \hookrightarrow S^3$ be a knot, and Σ_g a Seifert surface for K. The isomorphism $H_1(S^3 \backslash \Sigma_g;\mathbb{Z}) \longrightarrow H^1(\Sigma_g;\mathbb{Z}) = \mathrm{Hom}(H_1(\Sigma_g;\mathbb{Z}),\mathbb{Z})$ produced by Alexander duality is given by the linking pairing, i.e.:*
$$\ell k(-,-) \colon H_1(S^3 \backslash \Sigma_g;\mathbb{Z}) \times H_1(\Sigma_g) \longrightarrow \mathbb{Z}.$$

Proof. Omitted. The proof amounts to unravelling the definition of the isomorphism provided by Alexander duality. □

4. The Alexander Polynomial

In this section we will construct the Alexander polynomial of a knot.

4.1. Glimpses of Algebra

Before constructing the Alexander polynomial we first take a quick foray into algebra, in order to define the *order ideal*, which we need for our construction of the Alexander polynomial. All rings in this project are commutative, unital rings. We start out small with the following result which we will use to show, abstractly, that $\mathcal{A}(K)$ is finitely presented in section 4.2.1.

Lemma 4.1.1. *The ring* Λ *is Noetherian.*

Proof. The ring $\mathbb{Z}$ is Noetherian (it is Euclidean) so, by Hilberts Basis Theorem [1, theorem 7.5], $\mathbb{Z}[t]$ is Noetherian and so $(\mathbb{Z}[t])_t \cong \mathbb{Z}[t^{\pm 1}] \cong \Lambda$ is Noetherian, by [1, prop. 7.3]. □

4.1.1. *Finite presentability.* In this section we will introduce finite presentability of modules.

Warning. In [7], Rolfsen uses *row*-vectors while we use *column*-vectors.

Notation. For a ring R we denote by $\mathrm{Mat}_R(n,m)$ the set of $n \times m$-matrices over R.

Definition 4.1.2. Let R be a ring and let M be an R-module.

i) We call M *finitely generated* (over R) if there exists $n \in \mathbb{N}$ and an exact sequence of the form
$$R^n \longrightarrow M \longrightarrow 0.$$

ii) We call M *finitely presented* (over R) if there exists $m, n \in \mathbb{N}$ and an exact sequence of the form
$$R^m \xrightarrow{\ p\ } R^n \longrightarrow M \longrightarrow 0.$$
In this case we call such a p a *presentation* of M. We will say that a matrix in $\mathrm{Mat}_R(n,m)$ a *presents* M if it represents a presentation of M. ○

Notation. Let $P \in \mathrm{Mat}_R(n,m)$ represent $p\colon R^m \longrightarrow R^n$. We will let $\operatorname{coker} P$ denote $\operatorname{coker} p$ (this does, of course, not depend on the representation of P up to isomorphism).

Remark. Note that any matrix P, presents $\operatorname{coker} P$ and that a matrix presents a module M, if and only if it presents all modules isomorphic to M. So a matrix P presents M if and only if $M \cong \operatorname{coker} P$.

Remark. For you category-theorists out there,[5] an equivalent definition of finitely presented would be that M is an "(ω-)compact object" in $\mathbf{Mod}_R$ or just a "finitely presented object" in $\mathbf{Mod}_R$ depending on name-convention.

Lemma 4.1.3. *A ring* R *is Noetherian if and only if being a finitely presented module over* R *is the same as being a finitely generated module over* R.

[5] you know who you are.

Proof. Being finitely presented implies being finitely generated over any ring since being finitely presented has a stricter requirement. We will first prove the "if" part of the statement. To do this it suffices to show that if R is a ring and there exists an R-module M, which is finitely generated over R, but not finitely presented over R, then R is not Noetherian. Fix such a finitely generated R-module M and fix a surjective map $f\colon R^n \twoheadrightarrow M$. Since M is not finitely presented, the sub-(R-)module $\ker f$ of R^n cannot be finitely generated (as we will see in more detail in the proof of the "only if" part of the statement). But this means, that R cannot be Noetherian (because if R is Noetherian it follows from [1, corollary 6.4] that R^n is Noetherian and by [1, prop. 6.2] that $\ker f$ is finitely generated over R).

We will now prove the "only if" part of the statement. To do this it suffices to show, that if R is Noetherian then being finitely generated over R implies being finitely presented over R. Assume R is Noetherian and let M be a finitely generated R-module. Since M is finitely generated we can fix a surjection $f\colon R^n \twoheadrightarrow M$ for some $n \in \mathbb{N}$. The identity $\mathrm{id}\colon R^n \longrightarrow R^n$ gives us that R^n is a finitely generated R-module so, since R is Noetherian, it follows from [1, prop. 6.5] that R^n is a Noetherian module. Because $\ker f$ forms a submodule of R^n it now follows from [1, prop. 6.2] that $\ker f$ is finitely generated. Because $\ker f$ is finitely generated we can fix $p\colon R^m \twoheadrightarrow \ker f$ for some $m \in \mathbb{N}$. Extending the codomain of p to R^n gives us the exact sequence

$$R^m \xrightarrow{\ p\ } R^n \xrightarrow{\ f\ } M \longrightarrow 0,$$

so M is finitely presented. □

4.1.2. *The order ideal.* In this section we will define the *order ideal*, also called the *(level 0) Fitting ideal*, of a finitely presented module.

Construction 4.1.4. Let M be a finitely presented R-module and $P \in \mathrm{Mat}_R(n, m)$ be a matrix presenting M. We define the *order ideal* of P, which we will denote by $\mathcal{O}_P$, to be the ideal in R generated by the $n \times n$-minors of P with the convention, that if $m < n$ then $\mathcal{O}_P = (0)$.

Theorem 4.1.5. *Two matrices P, Q over R present isomorphic R-modules if and only if it is possible to get from one to the other by applying the following operations (also found on [7, p. 204] - with row-vector convention) finitely many times. Furthermore, the order ideal of a matrix is unchanged by each of these operations.*

- **(1)** *Interchange two rows or columns.*
- **(2)** *Add to any row an R-linear combination of other rows.*
- **(3)** *Add to any column an R-linear combination of other columns.*
- **(4)** *Multiply a row or column by a unit of R.*
- **(5)** *Replace P with the matrix*

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline r_1 & & & \\ r_2 & & P & \\ \vdots & & & \end{array}\right]$$

  with $r_i \in R$.
- **(6)** *The reverse of* **(5)***.*
- **(7)** *Adjoin a new column which is an R-linear combination of columns of P.*
- **(8)** *Delete a column which is an R-linear combination of columns of P.*

We apologise sincerely, dear reader; some parts of the following proof is horribly written. We recommend the reader proof this result themselves as this is (probably) easier and more enlightening (and less painful). Otherwise see [8, Section 07Z6] for another proof.

Proof. We begin by proving the "if" part of the statement, i.e. that applying each of the operations does not change $\operatorname{coker} P$ up to isomorphism. The operations **(1)**–**(4)** correspond to change of basis, which does not change $\operatorname{coker} P$ up to isomorphism.

We will now prove that operation **(5)** (and thereby also **(6)**) does not change $\operatorname{coker} P$ up to isomorphism. Let

$$P' = \begin{bmatrix} 1 & 0 \\ r & P \end{bmatrix}$$

be the result of applying **(5)** to P. Define $d_0 \colon R^{n+1} \longrightarrow R^n$ by

$$\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \overset{d_0}{\longmapsto} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - x_0 r, \qquad \text{for all } \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \in R^{n+1}.$$

We see that $x \in \operatorname{im} P'$ implies $d_0(x) \in \operatorname{im} P$ for all $x \in R^{n+1}$, so we can define $\phi\colon \operatorname{coker} P' \longrightarrow \operatorname{coker} P$ as the unique map making the following diagram commute:

$$\begin{array}{ccc} R^{n+1} & \xrightarrow{\ d_0\ } & R^n \\ \downarrow & & \downarrow \\ \operatorname{coker} P' & \xrightarrow[\exists! \phi]{} & \operatorname{coker} P, \end{array} \tag{4.1}$$

where the unnamed maps are the projections. We will now show, that ϕ is an isomorphism. The projections and d_0 in (4.1) is surjective, so ϕ must be surjective as well. We will now show that ϕ is injective, which is equivalent to $\ker \phi = 0$, which is equivalent to: If $x = [x_0, \ldots, x_n]^\dagger \in R^{n+1}$ such that $d_0(x) \in \operatorname{im} P$ then $x \in \operatorname{im} P'$. Let $x = [x_0, \ldots, x_n]^\dagger \in R^{n+1}$ such that $d_0(x) \in \operatorname{im} P$. Fix $y = [y_1, \ldots, y_m]^\dagger \in R^m$ such that $Py = d_0(x) = x' - x_0 r$. Let x' denote $[x_1, \ldots, x_n]^\dagger$ and y' denote $[x_0, y_1, \ldots, y_m]^\dagger \in R^{m+1}$. We now see that

$$P'y' = \begin{bmatrix} 1 & 0 \\ r & P \end{bmatrix} \begin{bmatrix} x_0 \\ y \end{bmatrix} = x_0 \begin{bmatrix} 1 \\ r \end{bmatrix} + \begin{bmatrix} 0 \\ Py \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 r \end{bmatrix} + \begin{bmatrix} 0 \\ x' - x_0 r \end{bmatrix} = \begin{bmatrix} x_0 \\ x' \end{bmatrix} = x,$$

so $x \in \operatorname{im} P'$ and we are done.

We will now prove that operation **(7)** (and thereby also **(8)**) does not change $\operatorname{coker} P$ up to isomorphism. We have $\operatorname{cod} P' = \operatorname{cod} P$ and $\operatorname{im} P' = \operatorname{im} P$ (since the image is the span of the columns), so we have $\operatorname{coker} P' = \operatorname{coker} P$.

Now, we prove the "only if" part of the statement. Let P and Q be matrices over R presenting isomorphic R-modules M and N respectively. We now want to show that it is possible to get from P to Q using the operations finitely many times. We will do this by constructing a larger matrix S from P and Q and showing, that we can get P *and* Q from S by finitely many applications of the operations. Fix an isomorphism $\phi\colon M \longrightarrow N$. We note that the matrix

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

presents $M \oplus N$, so adding the relations $(m, \phi(m)) = 0$ for all $m \in M$ will give us a presentation of M (and, of course N). We can also add the *redundant* and equivalent relations $(\phi^{-1}(n), n) = 0$ for all $n \in N$, making our construction symmetric in P and Q. Let $g_1, \ldots, g_n$ and $g'_1, \ldots g'_{n'}$ denote the generators of M and N with respect to P and Q respectively. Since the g_i generate M and the g'_i generate N, we can write $\phi(g_i) = \sum_j c_{i,j} g'_j$ and $\phi^{-1}(g'_i) = \sum_j c'_{i,j} g_j$ and define $C \coloneqq [c_{i,j}]$ and $C' \coloneqq [c'_{i,j}]$. We can therefore add the relators $(g_i, \phi(g_i))$ and $(\phi^{-1}(g'_i), g'_i)$ by defining

$$S \coloneqq \begin{bmatrix} P & 0 & I & C' \\ 0 & Q & C & I \end{bmatrix}.$$

It now suffices to show, that we can get from S to both P and Q using the operations finitely many times, which is what we will do. Because of our added redundancy, the argument we give will be symmetric in P and Q, so we will only state how to get from S to Q. The redundancy allows us to use **(8)** to remove $\left[\begin{smallmatrix} C' \\ I \end{smallmatrix}\right]$ and get:

$$\begin{aligned} S := \begin{bmatrix} P & 0 & I & C' \\ 0 & Q & C & I \end{bmatrix} &\overset{(8)}{\rightsquigarrow} \begin{bmatrix} P & 0 & I \\ 0 & Q & C \end{bmatrix} \\ &\overset{(3)}{\rightsquigarrow} \begin{bmatrix} 0 & 0 & I \\ -CP & Q & C \end{bmatrix} \\ &\overset{(1)}{\rightsquigarrow} \begin{bmatrix} I & 0 & 0 \\ C & Q & -CP \end{bmatrix} \\ &\overset{(6)}{\rightsquigarrow} \begin{bmatrix} Q & -CP \end{bmatrix} \\ &\overset{(8)}{\rightsquigarrow} Q, \end{aligned}$$

where we, in the last step, use that $\operatorname{im} CP \subset \operatorname{im} Q$, which follows from commutativity and exactness of the rows of:

$$\begin{array}{ccccccc} R^m & \xrightarrow{P} & R^n & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle \phi} & & \\ R^{m'} & \xrightarrow{Q} & R^{n'} & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

This concludes the proof of the “only if” part of the statement.

We will now prove, that $\mathcal{O}_P$ is unchanged by the operations for $P \in \operatorname{Mat}_R(n, m)$. In the case $m < n$ this is trivial for all the operations except **(7)** and **(8)**, so we will consider these cases as treated. Applying **(1)** to P can only change the signs of the $n \times n$ minors of P and does thereby not change the ideal they generate, $\mathcal{O}_P$.

Denote by $[a]$ the set $\{1, \ldots, a\}$ with the well-ordering from $\mathbb{N}$ for all $a \in \mathbb{N}$ and let $\boldsymbol{\Delta}_+(a, b)$ denote the set of strictly increasing functions $[a] \longrightarrow [b]$ for all $a, b \in \mathbb{N}$. For a matrix $X = [x_1, \ldots, x_m] \in \operatorname{Mat}_R(n, m)$ denote by X_τ the $n \times n$ submatrix $[x_{\tau(1)}, \ldots, x_{\tau(n)}] \in \operatorname{Mat}_R(n, n)$ of X and denote by $\mathfrak{m}_\tau$ the determinant $\det P_\tau$ for all $\tau \in \boldsymbol{\Delta}_+(n, m)$. Clearly $\tau \mapsto X_\tau$ is a bijection from $\boldsymbol{\Delta}_+(n, m)$ to the set of $n \times n$ submatrices of X for any $X \in \operatorname{Mat}_R(n, m)$. Therefore $\mathcal{O}_P = (\{\mathfrak{m}_\tau \mid \tau \in \boldsymbol{\Delta}_+(n, m)\})$.

We will now prove that **(3)** does not change $\mathcal{O}_P$. Let $P =: [p_1, \ldots, p_m]$. Now, fix an i and apply **(3)** to P at the i^{th} column to get $P' := [p_1, \ldots, p_i + \sum_{j \neq i} a_j p_j, \ldots, p_m]$. We now want to show, that $\mathcal{O}_P = \mathcal{O}_{P'}$. Let $\sigma \in \boldsymbol{\Delta}_+(n, m)$. If $i \notin \operatorname{im} \sigma$ we have $P'_\sigma = P_\sigma$ and so $\det P'_\sigma = \mathfrak{m}_\sigma$. If $i \in \operatorname{im} \sigma$ there exists a (unique) $k \in [n]$ such that $\sigma(k) = i$ and then, letting $P' =: [p'_1, \ldots, p'_m]$, we see that

$$\begin{aligned} \det P'_\sigma &= \det[p'_{\sigma(1)}, \ldots, p'_{\sigma(k)}, \ldots, p'_{\sigma(m)}] \\ &= \det[p_{\sigma(1)}, \ldots, p_i + \sum_{j \neq i} a_j p_j, \ldots, p_{\sigma(m)}] \\ &= \det[p_{\sigma(1)}, \ldots, p_{\sigma(m)}] + \sum_{j \neq i} a_j \det[p_{\sigma(1)}, \ldots, p_j, \ldots, p_{\sigma(m)}] \\ &= \mathfrak{m}_\sigma + \sum_{j \neq i} a_j \det[p_{\sigma(1)}, \ldots, p_j, \ldots, p_{\sigma(n)}] \\ &= \mathfrak{m}_\sigma + \sum_{j \neq i} a'_j \mathfrak{m}_{\sigma_j}, \end{aligned}$$

where σ_j is the unique element of $\boldsymbol{\Delta}_+(n, m)$ such that $\operatorname{im} \sigma_j = (\operatorname{im} \sigma \backslash \{i\}) \cup \{j\}$ (so, in particular $i \notin \operatorname{im} \sigma_j$ for $j \neq i$) and a'_j is a_j if $\sigma_j(k) = j$ and $-a_j$ otherwise for all $j \neq i$. Now, it follows, that $\mathcal{O}_{P'} = \mathcal{O}_P$ because the generators of $\mathcal{O}_{P'}$ have only been changed in the following way, which does not change the ideal, they generate: We have added R-linear combinations of $\mathfrak{m}_\tau$'s with $i \notin \operatorname{im}\tau$ to $\mathfrak{m}_\sigma$'s with $i \in \operatorname{im}\sigma$.

Applying **(2)** or **(4)** to P either does nothing or the same to every $n \times n$ submatrix of P so, since the determinant is independent (up to R-associatedness) of these, the order ideal is unchanged by these.

We will now prove, that operation **(5)** does not change $\mathcal{O}_P$. Again, apply **(5)** to P and denote the resulting $(n+1) \times (m+1)$ matrix by P'. The top row of any $(n+1) \times (n+1)$ submatrix of P' not containing the left-most column of P' is identically 0 and so the determinant of such a submatrix is 0. Therefore we can disregard these submatrices (as they do not contribute to $\mathcal{O}_{P'}$) and they are exactly the P'_τ with $1 \notin \operatorname{im}\tau$. The rest are exactly the P'_τ with $\tau \in \mathbf{\Delta}_+(n+1, m+1)$ in the image of the function $s\colon \mathbf{\Delta}_+(n,m) \longrightarrow \mathbf{\Delta}_+(n+1, m+1)$ defined by

$$s(\sigma)(i) = \begin{cases} 1 & i = 1 \\ \sigma(i-1)+1 & \text{otherwise} \end{cases}$$

for all $\sigma \in \mathbf{\Delta}_+(n,m)$. Now, by Laplace expansion, we have $\det P'_{s(\tau)} = \det P_\tau = \mathfrak{m}_\tau$ so the operation has only added minors equal to 0 and hence not changed $\mathcal{O}_P$.

The proof, that operation **(7)** does not change $\mathcal{O}_P$ is similar enough to **(3)** that we will omit it (though one has to check it in the case $n < m$).

That $\mathcal{O}_P$ is independent of **(8)** and **(6)** follows by symmetry of independence of **(7)** and **(5)** respectively. □

Corollary 4.1.6. *Let M be a finitely presented R-module then the order ideal, $\mathcal{O}_P$, in R is independent of both the choice of matrix-representation, P, of p and the choice of presentation, p, of M.*

In light of this theorem and corollary we can now define the order ideal of M for any finitely presented module M.

Definition 4.1.7. Let M be a finitely presented R-module. The *order ideal*, denoted $\mathcal{O}_M$, of M is the order ideal of any matrix presenting M. ○

Remark. Note, that if there exists a square matrix presenting M, then the order ideal of M is principal.

4.2. Construction of the Alexander Polynomial

It turns out, that $\mathcal{A}(K)$ is a finitely presented Λ-module and that its order ideal is principal for all (smooth) knots $K\colon S^1 \hookrightarrow S^3$. We want to define the *Alexander polynomial* of a knot K as a generator of the order ideal of $\mathcal{A}(K)$. This is what we will do in this section. First, we will show, abstractly, that $\mathcal{A}(K)$ is finitely presented and then we will construct a square matrix presenting $\mathcal{A}(K)$.

4.2.1. *Abstract finite presentability argument.* In this section we will show, abstractly, that $\mathcal{A}(K)$ is finitely presented. In section 4.2.2 we construct a presentation of $\mathcal{A}(K)$ (not using the results of this section).

We can lift Δ-complex structures along coverings, i.e.:

Construction 4.2.1. Let $p\colon \tilde{X} \longrightarrow X$ be a path-connected covering and let $X_\bullet$ be a Δ-complex structure on X. Let $\sigma\colon \Delta^n \hookrightarrow X$ be an element of X_n. Now (since Δ^n is contractible) it follows from [3, theorem 2.7.2][6] that σ lifts, uniquely, to a map $\tilde{\sigma}_x\colon \Delta^n \hookrightarrow X$ along p for each $x \in \operatorname{fib}(p)$. Define $\tilde{X}_\bullet$ by $\tilde{X}_n \coloneqq \{\tilde{\sigma}_x \mid \sigma \in X_n, x \in \operatorname{fib}(p)\}$.

[6] Since X has a Δ-complex structure it is locally path-connected and therefore, since $p\colon \tilde{X} \longrightarrow X$ is a covering, so is $\tilde{X}$.

Lemma 4.2.2. *In construction 4.2.1, $\tilde{X}_\bullet$ is a Δ-complex structure on $\tilde{X}$.*

Proof. Readily follows. □

Lemma 4.2.3. *The Λ-module $\mathcal{A}(K)$ is finitely presented over Λ.*

Proof. Since X_K is a closed, compact manifold with boundary, it has the structure of a finite Δ-complex. Fix a finite Δ-complex structure on X_K and call it $X_\bullet$. By using lemma 4.2.2, we can now lift $X_\bullet$ to a Δ-complex structure $\tilde{X}_\bullet$ on $\tilde{X}_K$ along the covering $p\colon \tilde{X}_K \longrightarrow X_K$. The group $\mathrm{Deck}(p) \cong \mathbb{Z}$ acts on $\tilde{X}_\bullet$ by post-composition which gives an action of $\mathrm{Deck}(p)$ on $H_1^\Delta(\tilde{X})$. Under the isomorphism $H_1^\Delta(\tilde{X}) \cong H_1(\tilde{X};\mathbb{Z})$ induced by the inclusion $\tilde{X}_\bullet \hookrightarrow \mathbf{Sing}_\bullet^{\mathrm{nd}}(\tilde{X})$ these actions of $\mathrm{Deck}(p)$ (the other one being the one from $\mathcal{A}(K)$) agree. This means, $\Delta_1(\tilde{X}_K) \cong \Delta_1(X_K) \otimes \Lambda$ and so $\Delta_1(\tilde{X}_K)$ is free of finite rank over Λ (Which is Noetherian by lemma 4.1.1) so $\Delta_1(\tilde{X}_K)$ is Noetherian over Λ and therefore the (simplicial) 1-cycles and 1-boundaries are as well and then it follows from [1, prop. 6.3] that $\mathcal{A}(K)$ is Noetherian and hence finitely generated over Λ as well. Since Λ is Noetherian we are done by lemma 4.1.3. □

4.2.2. *Construction of the Alexander polynomial.* To show that $\mathcal{O}_{\mathcal{A}(K)}$ is principal we will describe the generators and relations of $\mathcal{A}(K)$ by giving a matrix presenting $\mathcal{A}(K)$. The matrix in question is $V - tV^\dagger$ for a Seifert matrix V, where we think of the matrix V as a matrix over Λ by extension of scalars.

Now, for the main result of this project:

Theorem 4.2.4. *Let $K\colon S^1 \hookrightarrow S^3$ be a knot and V a Seifert matrix for K, then $V^\dagger - tV$ presents $\mathcal{A}(K)$ over Λ.*

To prove this theorem we will use the Mayer–Vietoris sequence twice, first on X_K and then (using the results of the first application) on $\tilde{X}_K$. From this, we will extract a presentation of $\mathcal{A}(K)$, which we will then show gives us that $V - tV^\dagger$ presents $\mathcal{A}(K)$.

Proof. In this proof we will only use homology with integral coefficients, therefore we will suppress the coefficients from the notation.

First, we construct the two subspaces of X_K we will use the Mayer–Vietoris sequence on. Choose a tubular neighbourhood $\nu(K)$ of K then choose a Seifert surface Σ_g of K, which is "trivial" in $\nu(K)$. Now, choose a bicollar b of Σ_g such that $b(K \times [-1,1]) \subset \mathring{\nu}(K)$. As always, X_K denotes $S^3 \backslash \mathring{\nu}(K)$. Define

$$\mathbf{\Sigma} \coloneqq b(\Sigma_g \times [-1,1]) \cap X_K \quad \text{and} \quad X_{\mathbf{\Sigma}} \coloneqq X_K \backslash \mathring{b}(\Sigma_g \times [-1/2, 1/2]).$$

These will be the subspaces we will use Mayer–Vietoris on. Define

$$\mathbf{\Sigma}^+ \coloneqq b(\Sigma_g \times [1/2, 1]) \cap X_K \quad \text{and} \quad \mathbf{\Sigma}^- \coloneqq b(\Sigma_g \times [-1, -1/2]) \cap X_K.$$

Note that $\mathbf{\Sigma} \cap X_{\mathbf{\Sigma}} = \mathbf{\Sigma}^+ \cup \mathbf{\Sigma}^-$, the union being disjoint, and $\mathbf{\Sigma}^+ \simeq \Sigma_g^+ \simeq \Sigma_g^- \simeq \mathbf{\Sigma}^-$.

Let $p\colon \tilde{X}_K \longrightarrow X_K$ be the infinite cyclic covering and let $\tilde{\mathbf{\Sigma}} \coloneqq p^{-1}(\mathbf{\Sigma})$ and $\tilde{X}_{\mathbf{\Sigma}} \coloneqq p^{-1}(X_{\mathbf{\Sigma}})$. We can illustrate our current construction in the following diagram, labelling the inclusions:

$$\begin{array}{ccccc} \tilde{\mathbf{\Sigma}} & \overset{\tilde{\iota}^1}{\hookrightarrow} & \tilde{X}_K & \overset{\tilde{\iota}^2}{\hookleftarrow} & \tilde{X}_{\mathbf{\Sigma}} \\ \downarrow & \ulcorner & \downarrow p & \urcorner & \downarrow \\ \mathbf{\Sigma} & \overset{\iota^1}{\hookrightarrow} & X_K & \overset{\iota^2}{\hookleftarrow} & X_{\mathbf{\Sigma}}. \end{array} \tag{4.2}$$

We note that $\mathrm{int}(\mathbf{\Sigma}) \cup \mathrm{int}(X_{\mathbf{\Sigma}}) = X_K$, so we can use the Mayer–Vietoris sequence on the bottom row of (4.2), and get the exact sequence:

$$H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-) \longrightarrow H_1(\mathbf{\Sigma}) \oplus H_1(X_{\mathbf{\Sigma}}) \longrightarrow H_1(X_K) \xrightarrow{\delta} H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-). \tag{4.3}$$

We will now show that p is trivial over $\mathbf{\Sigma}$ and $X_{\mathbf{\Sigma}}$ using the following lemma:

Lemma 4.2.5. *Let $A \subset X_K$ be a subspace. If the map $H_1(A) \longrightarrow H_1(X_K)$ induced by the inclusion is 0 then p is trivial over A.*

Proof of lemma. Let $i\colon A \hookrightarrow X_K$ denote the inclusion. Using [3, Theorem 2.7.2], we get that p is trivial over A if and only if $i_*\pi_1(A, a) \subset p_*\pi_1(\tilde{X}_K, \tilde{a})$ for all $a \in A$ and all $\tilde{a} \in \mathrm{fib}_p(a)$. By definition of p we have $p_*\pi_1(\tilde{X}_K, \tilde{a})$ is the commutator subgroup of $\pi_1(X_K, a)$ which is exactly $\ker(\hbar\colon \pi_1(X_K, a) \longrightarrow H_1(X_K))$. Therefore we have that p is trivial over A if and only if the composition

$$\pi_1(A, a) \xrightarrow{i_*} \pi_1(X_K, a) \xrightarrow{\hbar} H_1(X_K)$$

is 0. We have the commutative square

$$\begin{array}{ccc} \pi_1(A, a) & \xrightarrow{i_*} & \pi_1(X_K, a) \\ \hbar\downarrow & & \hbar\downarrow \\ H_1(A) & \xrightarrow{i_*} & H_1(X_K), \end{array}$$

from which it follows, that if $i_*\colon H_1(A) \longrightarrow H_1(X_K)$ is 0, then

$$\hbar \circ i_* = i_* \circ \hbar = 0 \circ \hbar = 0$$

and so p is trivial over A. $\square_{\text{lemma}}$

To show, that p is trivial over $\mathbf{\Sigma}$ and $X_{\mathbf{\Sigma}}$ it suffices (by the above lemma) to show that the maps induced on homology by the inclusions $\mathbf{\Sigma} \hookrightarrow X_K$ and $X_{\mathbf{\Sigma}} \hookrightarrow X_K$ are trivial. We will denote the inclusions as follows:

$$\begin{array}{ccc} \mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^- & \xhookrightarrow{i^2} & X_{\mathbf{\Sigma}} \\ i^1\downarrow & & \downarrow\iota^2 \\ \mathbf{\Sigma} & \xhookrightarrow{\iota^1} & X_K \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{\mathbf{\Sigma}}^+ \sqcup \tilde{\mathbf{\Sigma}}^- & \xhookrightarrow{\tilde{i}^2} & \tilde{X}_{\mathbf{\Sigma}} \\ \tilde{i}^1\downarrow & & \downarrow\tilde{\iota}^2 \\ \tilde{\mathbf{\Sigma}} & \xhookrightarrow{\tilde{\iota}^1} & \tilde{X}_K. \end{array}$$

The map $\iota^1_*\colon H_1(\mathbf{\Sigma}) \longrightarrow H_1(X_K)$ is trivial for geometric reasons (none of the generators form a meridian of K), so p is trivial over $\mathbf{\Sigma}$.

To show that $\iota^2_*\colon H_1(X_{\mathbf{\Sigma}}) \longrightarrow H_1(X_K)$ is 0, it now suffices to show that

$$\iota^1_* + \iota^2_*\colon H_1(\mathbf{\Sigma}) \oplus H_1(X_{\mathbf{\Sigma}}) \longrightarrow H_1(X_K)$$

is 0, and (since this is the map in (4.3)) showing this is 0 is equivalent to showing that δ in (4.3) is injective (by exactness). To show that δ in (4.3) is injective we will "calculate" it. We have that $H_1(X_K) \cong \mathbb{Z}$ generated by the meridian and that $H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-) \cong H_0(\mathbf{\Sigma}^+) \oplus H_0(\mathbf{\Sigma}^-) \cong \mathbb{Z}^2$. Define $\mathcal{U} := \{\mathbf{\Sigma}, X_{\mathbf{\Sigma}}\}$. Recall that δ is a connecting homomorphism in homology of the short exact sequence

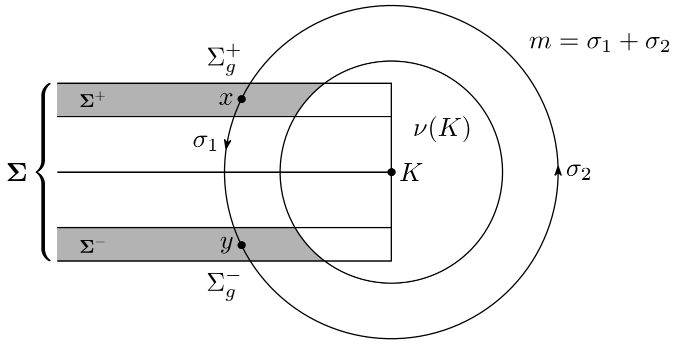
$$0 \longrightarrow C_*(\mathbf{\Sigma} \cap X_{\mathbf{\Sigma}}) \longrightarrow C_*(\mathbf{\Sigma}) \oplus C_*(X_{\mathbf{\Sigma}}) \longrightarrow C^{\mathcal{U}}_*(X_K) \longrightarrow 0$$

of chain complexes. To calculate δ we simply need to calculate where it sends 1 i.e. the meridian (up to orientation/sign). Pick a meridian $m \in H_1(X_K)$.[7] Define

[7] We can pick a specific model for the meridian inside a tubular neighbourhood of K allowing us to control the situation completely.

$\sigma_1\colon \Delta^1 \longrightarrow \mathrm{int}(\mathbf{\Sigma})$ and $\sigma_2\colon \Delta^1 \longrightarrow \mathrm{int}(X_{\mathbf{\Sigma}})$ as in fig. 4. Now, $\sigma_1 + \sigma_2$ is a $\mathcal{U}$-

FIGURE 4. Cross-section of the tubular neighbourhood of K. The grey area shows the intersection $\mathbf{\Sigma} \cap X_{\mathbf{\Sigma}}$.



small 1-chain and $(\iota_*^1 + \iota_*^2)(\sigma_1, \sigma_2) = m$. Therefore, by definition of the connecting homomorphism in homology, the meridian is sent to the (unique) homology class of the 0-cycle whose image under $(i_*^1, -i_*^2)$ is $\partial(\sigma_1, \sigma_2)$ in $C_1(\mathbf{\Sigma}) \oplus C_1(X_{\mathbf{\Sigma}})$. We see that $\partial(\sigma_1, \sigma_2) = (y - x, x - y) = (i_*^1, -i_*^2)(y - x)$ and so $\delta([m]) = [x - y]$. This means $\delta\colon H_1(X_K) \longrightarrow H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)$ corresponds to the map $[1, -1]^\dagger\colon \mathbb{Z} \longrightarrow \mathbb{Z}^2$, which is injective, so δ is injective.

Since the inclusions in the top row of (4.2) commute with the deck transformations of p, the Mayer–Vietoris sequence of the top row of (4.2) gives us a long exact sequence of Λ-modules. Since p is trivial over $\mathbf{\Sigma}$ and $X_{\mathbf{\Sigma}}$ we get that $H_n(\tilde{\mathbf{\Sigma}}) \cong H_n(\mathbf{\Sigma})[t^{\pm 1}]$ and $H_n(\tilde{X}_{\mathbf{\Sigma}}) \cong H_n(X_{\mathbf{\Sigma}})[t^{\pm 1}]$ as Λ-modules for all $n \in \mathbb{Z}$. Therefore, the Mayer–Vietoris sequence of the top row of (4.2) gives us the exact sequence:

$$H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \xrightarrow{(\tilde{i}_*^1, -\tilde{i}_*^2)} H_1(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_1(X_{\mathbf{\Sigma}})[t^{\pm 1}] \xrightarrow{\tilde{\iota}_*^1 + \tilde{\iota}_*^2} H_1(\tilde{X}_K) \xrightarrow{\tilde{\delta}} H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \xrightarrow{f} H_0(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_0(X_{\mathbf{\Sigma}})[t^{\pm 1}] \tag{4.4}$$

of Λ-modules.

We will first show that $\tilde{\delta}$ in (4.4) is 0. We will do this by showing, that f is injective which we will do by “calculating” it. We have that

$$H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \cong (\mathbb{Z} \oplus \mathbb{Z})[t^{\pm 1}] \cong \Lambda \oplus \Lambda$$

and, similarly, that

$$H_0(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_0(X_{\mathbf{\Sigma}})[t^{\pm 1}] \cong \Lambda \oplus \Lambda.$$

Choose a point $x \in p^{-1}(\mathbf{\Sigma}^+)$. Then $[x]$ generates $H_0(\mathbf{\Sigma})[t^{\pm 1}]$ and $H_0(X_{\mathbf{\Sigma}})[t^{\pm 1}]$ and we can define isomorphisms $\phi\colon H_0(\mathbf{\Sigma})[t^{\pm 1}] \longrightarrow \Lambda$ and $\phi'\colon H_0(X_{\mathbf{\Sigma}}) \longrightarrow \Lambda$ by $[x] \mapsto 1$. Similarly we can choose a point $y \in p^{-1}(\mathbf{\Sigma}^-)$ such that x and y are in the same path component (lift) of $\tilde{\mathbf{\Sigma}}$. Now, by the above defined isomorphism we have $\phi([y]) = 1$. We know that $[y]$ also generates $H_0(X_{\mathbf{\Sigma}})[t^{\pm 1}]$ and so $\phi'([y]) \in \Lambda^\times$ i.e. an element of the form $\pm t^n$ with $n \in \mathbb{Z}$. If $\phi'([y]) = \pm t^n$ with $n \neq \pm 1$ then $\tilde{X}_K$ is not path-connected, which contradictions p being the infinite cyclic covering. Therefore we can assume, WLOG, that $\phi'([y]) = t$. We have now calculated f, it is the unique map making

$$\begin{array}{ccc} H_0(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] & \xrightarrow{f} & H_0(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_0(X_{\mathbf{\Sigma}})[t^{\pm 1}] \\ \psi \downarrow & & \downarrow (\phi, \phi') \\ \Lambda \oplus \Lambda & \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ 1 & t \end{smallmatrix}\right]} & \Lambda \oplus \Lambda \end{array}$$

commute, where ψ is the isomorphism defined by $[x] \mapsto [1, 0]^\dagger$ and $[y] \mapsto [0, 1]^\dagger$. From this it is clear, that f is injective.

This means, we can extract the following exact sequence from (4.4):

$$H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \longrightarrow H_1(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_1(X_{\mathbf{\Sigma}})[t^{\pm 1}] \longrightarrow H_1(\tilde{X}_K) \longrightarrow 0,$$

so since $H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \cong H_1(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_1(X_{\mathbf{\Sigma}})[t^{\pm 1}] \cong \Lambda^{4g}$ and $H_1(\tilde{X}_K) = \mathcal{A}(K)$ we get a presentation of $\mathcal{A}(K)$. We will now calculate this presentation

$$H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-)[t^{\pm 1}] \longrightarrow H_1(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_1(X_{\mathbf{\Sigma}})[t^{\pm 1}]$$

of $\mathcal{A}(K)$ in (4.4) by calculating the corresponding map in (4.3) and use the same argument as when calculating f above to calculate the lift. Consider

$$H_1(\mathbf{\Sigma}^+ \sqcup \mathbf{\Sigma}^-) \longrightarrow H_1(\mathbf{\Sigma}) \oplus H_1(X_{\mathbf{\Sigma}}).$$

Pick a basis $([b_i])$ for $H_1(\Sigma_g) \cong H_1(\mathbf{\Sigma})$ and assume WLOG that V is a Seifert matrix for K with respect to $([b_i])$ i.e. $v_{i,j} = \ell k(b_i, b_j^+)$. Use $([b_i^+])$ and $([b_i^-])$ as bases for $H_1(\mathbf{\Sigma}^+)$ and $H_1(\mathbf{\Sigma}^-)$ respectively. With our chosen bases the map

$$H_1(\mathbf{\Sigma}^+) \oplus H_1(\mathbf{\Sigma}^-) \longrightarrow H_1(\mathbf{\Sigma})$$

is represented by $\begin{bmatrix} I & I \end{bmatrix}$. Using Alexander duality, and lemma 3.3.4 we can pick a basis (β_i) for $H^1(\mathbf{\Sigma}) \cong H_1(X_{\mathbf{\Sigma}})$ which is dual to $([b_i])$ with respect to the linking pairing i.e.:

$$\ell k(y, -) = \sum_i \ell k(y, b_i)\beta_i$$

for any $[y] \in H_1(X_{\mathbf{\Sigma}})$. We can now see that the maps $H_1(\mathbf{\Sigma}^+) \longrightarrow H^1(\mathbf{\Sigma})$ and $H_1(\mathbf{\Sigma}^-) \longrightarrow H^1(\mathbf{\Sigma})$ are given by

$$[b_i^+] \longmapsto \ell k(b_i^+, -) = \sum_j \ell k(b_i^+, b_j)\beta_j = \sum_j \ell k(b_j, b_i^+)\beta_j = \sum_j v_{j,i}\beta_j$$

and

$$[b_i^-] \longmapsto \ell k(b_i^-, -) = \sum_j \ell k(b_i^-, b_j)\beta_j = \sum_j \ell k(b_i, b_j^+)\beta_j = \sum_j v_{i,j}\beta_j$$

respectively, where we have used lemma 3.3.2. We have now shown that the map

$$H_1(\mathbf{\Sigma}^+) \oplus H_1(\mathbf{\Sigma}^-) \longrightarrow H_1(\mathbf{\Sigma}) \oplus H_1(X_{\mathbf{\Sigma}}),$$

with our choice of bases, is given by

$$\begin{bmatrix} I & I \\ V^\dagger & V \end{bmatrix}.$$

Using the previous argument, we can lift these bases to bases such that the map

$$H_1(\mathbf{\Sigma}^+)[t^{\pm 1}] \oplus H_1(\mathbf{\Sigma}^-)[t^{\pm 1}] \longrightarrow H_1(\mathbf{\Sigma})[t^{\pm 1}] \oplus H_1(X_{\mathbf{\Sigma}})[t^{\pm 1}]$$

from (4.4) is the map

$$\begin{bmatrix} I & I \\ V^\dagger & tV \end{bmatrix},$$

which presents $\mathcal{A}(K)$. We can now use the operations from theorem 4.1.5 to get:

$$\begin{bmatrix} I & I \\ V^\dagger & tV \end{bmatrix} \overset{(2)}{\rightsquigarrow} \begin{bmatrix} I & 0 \\ V^\dagger & tV - V^\dagger \end{bmatrix} \overset{(4)}{\rightsquigarrow} \begin{bmatrix} I & 0 \\ V^\dagger & V^\dagger - tV \end{bmatrix} \overset{(6)}{\rightsquigarrow} V^\dagger - tV,$$

so $V^\dagger - tV$ presents $\mathcal{A}(K)$ and we win! □

Corollary 4.2.6. *For any knot* $K\colon S^1 \hookrightarrow S^3$, *the order ideal* $\mathcal{O}_{\mathcal{A}(K)}$ *of* $\mathcal{A}(K)$ *is principal.*

Proof. The matrix $V^\dagger - tV$ is square. □

Remark. Morally, this not a corollary, but a *porism* (it follows from the *proof*, not the *statement*, of the theorem) – we get a square presentation of $\mathcal{A}(K)$ directly from the Mayer–Vietoris sequence after we have showed, that $\tilde{\delta}$ in (4.4) is 0.

Now, for the crowning jewel of this project:

Definition 4.2.7. Any generator of $\mathcal{O}_{\mathcal{A}(K)}$ is called the *Alexander polynomial* of K and is denoted by $\Delta_K(t)$. ◦

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